Qualitative analysis of causal cosmological models

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(Received 9 November 1995; accepted for publication 12 February 1996)

The Einstein's field equations of Friedmann–Robertson–Walker universes filled with a dissipative fluid described by both the *truncated* and *non-truncated* causal transport equations are analyzed using techniques from dynamical systems theory. The equations of state, as well as the phase space, are different from those used in the recent literature. In the de Sitter expansion both the hydrodynamic approximation and the non-thermalizing condition can be fulfilled simultaneously. For $\Lambda = 0$ these expansions turn out to be stable provided a certain parameter of the fluid is lower than 1/2. The more general case $\Lambda > 0$ is studied in detail as well. © *1996 American Institute of Physics.* [S0022-2488(96)03205-6]

I. INTRODUCTION

Recently, isotropic spatially homogeneous viscous cosmological models have been investigated using the causal (truncated and nontruncated) Israel–Stewart theory of irreversible processes, to modelize the bulk viscous transport.[1–3] It is known that dissipative processes may play a crucial role in the evolution of relativistic fluids both in cosmology and in high-energy astrophysical phenomena. The most oftenly used theory to describe such irreversible processes has been long since the first-order non-causal Eckart's theory[4] which however suffers from serious pathologies and drawbacks, i.e., superluminal velocities and instabilities.[5,6] In the late sixties Müller[7] proposed a second order theory in which the entropy flow depended on the dissipative variables besides the equilibrium ones. Israel and Stewart[8,9] and Pavón *et al.*[10] developed a fully relativistic formulation on that basis, the so-called *extended* or *transient* thermodynamics (see Ref. 11 for a recent and comprehensive review of the state of the art).

Shortly after Israel's paper appeared, Belinskii *et al.*[12] applied it to a viscous cosmological fluid using the so-called *truncated* version, in which some divergence terms in the transport equations were neglected. Most of the papers dealing with viscous and/or heat conducting cosmological models make use of such a truncated transport equation without stating clearly what the implications of such a simplification may be. Recently, some effort has been invested in analyzing to what extent the neglecting of the divergence terms can be justified from a physical point of view.[3,13] As far as we know, Hiscock and Salmonson[14] were the first to raise this point in the cosmological context. These authors stressed the key importance of the usually neglected divergence terms when obtaining viscosity-driven inflationary solutions. However, it is now clear that to get realistic solutions to the Einstein's field equations, the role played by the equations of state relating the different thermodynamic quantities is crucial. Hence the claim in Ref. 14 applies only to a Boltzmann gas.[15] In fact, the difficulty in using the extended transport equations lies mainly in the occurrence of some additional unknown coefficients, whose explicit expressions must be obtained from techniques other than those coming from thermodynamics, either kinetic or fluctuation theory,[16] more than in their intrinsic complexity.

Few exact solutions have been found to the Einstein's field equations with a non-perfect fluid described by extended thermodynamics[17,18] (ET for short). However, they were obtained under severe restrictions on the values for the free parameters in the transport equations. Obviously, any further attempt to get a deeper insight on the possible behavior of the solutions must rely on an approximate analysis of the equations. In this paper, we apply qualitative analysis techniques to the study of causal viscous Friedmann–Robertson–Walker (FRW) models with and without a

0022-2488/96/37(6)/2906/14/$10.00

positive cosmological constant. It is organized as follows. In section II we state the basic equations governing the models and discuss the equations of state to be used. In section III we apply the truncated version of ET whereas in section IV a corresponding analysis is carried out using the *full* version. In both cases a null and a positive cosmological constant are considered in turn. In section V we explore their dynamical consequences, and finally in section VI we summarize the main conclusions of the paper.

II. BASIC EQUATIONS

We restrict ourselves to a FRW space-time filled with a bulk viscous fluid and a positive cosmological constant Λ. The stress-energy tensor is

$$T_{ab}=(\rho+p+\Pi)u_a u_b+(p+\Pi-\Lambda)g_{ab}, \tag{1}$$

where u_a is the four velocity, ρ the energy density, p the equilibrium pressure, Π the bulk viscous pressure. Einstein's field equations for the spatially flat case (the only one we adress in this paper) are

$$H^2=\frac{\kappa}{3}\rho+\frac{\Lambda}{3}, \quad 3(\dot{H}+H^2)=-\frac{\kappa}{2}(\rho+3P_{eff})+\Lambda, \tag{2}$$

where $H\equiv\dot{R}/R$ is the Hubble factor, $R(t)$ the cosmic scale factor of the Robertson–Walker metric, $P_{eff}=p+\Pi$ and $\kappa=8\pi G/c^4$. An overdot denotes differentiation respect to time t. We assume the fluid obeys equations of state of the form

$$\zeta=\alpha\rho^m, \quad p=(\gamma-1)\rho, \quad \tau=\frac{\zeta}{\rho}, \tag{3}$$

where α is a positive constant, and γ the adiabatic index lying in the range $1<\gamma<2$ as the sound velocity $v_s/c=\gamma-1$ in the fluid must be lower than the speed of light. $\tau(\geqslant 0)$ is the relaxation time for transient bulk viscous effects, i.e., the time the system takes in going back to equilibrium once the divergence of the four-velocity has been switched off. The causal evolution equation for bulk viscous pressure can be cast into the form[3]

$$\Pi+\tau\dot{\Pi}=-3\zeta H-\frac{b}{2}\tau\Pi\left(3H+\frac{\dot{\tau}}{\tau}-\frac{\dot{T}}{T}-\frac{\dot{\zeta}}{\zeta}\right), \tag{4}$$

where $b=0$ for the truncated theory and $b=1$ for the full one. Since a dissipative expansion is non-thermalizing, the relaxation time must exceed the expansion rate H^{-1}. This leads to

$$\tau^{-1}<H, \tag{5}$$

which is a condition that reduces the interval of values of γ for which the model holds. As we shall see this restriction may be violated in the truncated theory as well as in the full theory when an ideal gas equation of state is assumed. This conflict can be circumvented by resorting to the expression for the speed of the viscous signals $a\equiv v^2/c^2\sim\zeta/\tau\rho$, which roughly implies[3,13]

$$\tau=\frac{\zeta}{a\rho}, \qquad 0<a<1, \tag{6}$$

and using (6) instead of (3c).

Most of the stability analysis below will be carried out for the de Sitter solutions, $H=$ const. As the universe undergoes a de Sitter expansion it could be argued that the hydrodynamic description (absolutely needed in our study if the results are to hold) might break down. In order for the fluid approach remains valid the mean collision time t_{col} must be less (in fact much less) than the expansion rate, i.e., $t_{col} < H^{-1}$. From kinetic theory one has $t_{col} = 1/n\sigma$ where n is the particle number density and σ the cross section for collisions. In general $\sigma(T)$ is an increasing function of the temperature whereas, for a FRW universe, $n \propto 1/R^3 (= e^{-3H_0 t}$ for a de Sitter universe). From equations (2a), (3a) and (6) the non-thermalizing condition (5) and the condition for the hydrodynamic approximation imply

$$\frac{e^{3H_0 t}}{n_0 \sigma} < H_0^{-1} < \frac{\alpha}{a} \left(\frac{3}{\kappa} \right)^{m-1} H_0^{2m-2}, \tag{7}$$

where n_0 is a positive but otherwise arbitrary integration constant. Later it will be shown that the second inequality in (7) can be fulfilled when suitable values for the arbitrary parameters are chosen. Moreover, the first inequality may hold for sufficiently early times (when the inflation era supposedly took place). As the temperature remains constant during this period the cross section σ can be taken approximately constant.[19]

Recently in performing the qualitative analysis of imperfect fluid cosmological models (see for instance Refs. 1,2,20,21) dimensionless equations of state were used in terms of the dimensionless variables x and y, defined as

$$x \equiv 3\rho/\Theta^2, \qquad y \equiv 9\Pi/\Theta^2. \tag{8}$$

The equations of state [Eq. (3.4a), (3.4b) in Ref. 20)]

$$p/\Theta^2 = p_0 x^l, \qquad \zeta/\Theta = \zeta_0 x^m, \tag{9}$$

with $\Theta(\equiv 3H)$ the expansion factor, coincide with (3a), (3b) only for $l = m = 1/2$. Furthermore, for the spatially flat FRW metric with $\Lambda = 0$, the case we are interested in, we have $x = 1$. Then the bulk viscous coefficient ζ varies as Θ irrespective of m, which restricts (9) to just one case: $m = 1/2$ in (3a). In this paper we shall consider only the *spatially flat* case $(k = 0)$ which allows us to take $(\dot{H}, H)$ as suitable dynamical variables in the phase space. In this case it appears to be more natural, especially when the above comments are taking into account, to adopt the oftenly used equations of state (3) rather than (9) in order to be able to compare our results with those in the literature for $m = 1/2$. Consequently, all the fixed points to be analyzed will correspond to either de Sitter or static spacetimes $(X \equiv \dot{H} = 0)$ the former being physically relevant in inflationary models. If one is interested in studying non-flat FRW models, the variables $(\dot{H}, H)$ become no longer appropriate and an approach similar to that of Coley[20] should by adopted.

III. QUALITATIVE ANALYSIS USING THE TRUNCATED THEORY

From equations (2), (3) and the expression (6) for τ we find for the Hubble factor the equation

$$\ddot{H} + 3\gamma H \dot{H} + \frac{a}{\delta} \left(H^2 - \frac{\Lambda}{3} \right)^{1-m} \dot{H} + \frac{(3H^2 - \Lambda)a}{2} \left[\frac{\gamma}{\delta} \left(H^2 - \frac{\Lambda}{3} \right)^{1-m} - 3H \right] = 0, \tag{10}$$

where $\delta = \alpha(3/\kappa)^{m-1}$. Equation (10) can be recast into the form

$$\dot{H} = P(H,X), \quad \dot{X} = Q(H,X), \tag{11}$$

where

$$P(H,X) = X, \tag{12}$$

$$Q(H,X)=-3\gamma HX-\frac{a}{\delta}\left(H^2-\frac{\Lambda}{3}\right)^{1-m}X-\frac{(3H^2-\Lambda)a}{2}\left[\frac{\gamma}{\delta}\left(H^2-\frac{\Lambda}{3}\right)^{1-m}-3H\right]. \tag{13}$$

The qualitative analysis begins by linearizing the system (11) for small perturbations–where the linear theory holds. Then the Jacobian matrix

$$\mathbf{L}=\begin{pmatrix} P_H & P_X \\ Q_H & Q_X \end{pmatrix}, \text{ with } P_H\equiv\frac{\partial P}{\partial H}, \text{ etc.,} \tag{14}$$

can be constructed. The elements of this matrix must be evaluated at the equilibrium points (h_i,X_i) (de Sitter and static solutions) which are found by solving the system $P(h_i,X_i)=Q(h_i,X_i)=0$. After diagonalizing $\mathbf{L}$ and obtaining its eigenvalues we can decide about the type of fixed points and their stability.

The analysis of the system (12), (13) for the two cases with $\Lambda=0$ and $\Lambda>0$ will be carried out in turn.

i. $\Lambda=0$

- $m=1/2$

We have the trivial fixed point,

$$(0,0),$$

which corresponds to an unstable static model. However, this case does not make sense as, by Einstein's equation (2a), $\rho=0$. If γ and δ fulfill the restriction

$$\gamma/\delta=3, \tag{15}$$

there exists an infinity of fixed points $(h_0,0)$, where h_0 denotes an arbitrary positive real constant. In this case the fixed points are parallel stable straight lines.

- $m\neq 1/2$

In the intervals $0\leqslant m<1/2$, $1/2<m<2$ there are two fixed points,

$$(0,0), \qquad (h_0,0), \tag{16}$$

with

$$h_0=\left(\frac{3\delta}{\gamma}\right)^{1/(1-2m)}, \tag{17}$$

whereas for $m\geqslant 2$ there is only one fixed point $(h_0,0)$. The discussion for the point (0,0) mimics that for $m=1/2$.

Let us define the auxiliar parameter,

$$\Sigma_1=\frac{1}{4a}\left(\gamma+\frac{a}{\gamma}\right)^2.$$

For $m<\frac{1}{2}-\Sigma_1$ the equilibrium point $(h_0,0)$ is an asymptotically stable focus, for $m=\frac{1}{2}-\Sigma_1$ it is an asymptotically stable degenerate node, whereas for $m>\frac{1}{2}-\Sigma_1$ two cases arise. If $\frac{1}{2}-\Sigma_1<m<\frac{1}{2}$, then the equilibrium point is an asymptotically stable node, whereas if $m>1/2$ it is a unstable saddle point.

In the paper by Pavón *et al.*[22] slightly different techniques were used to analyze the case $m=1/2$ and $a=1$. Their relevant parameter was our Σ_1 with $a=1$. Our results agree with those of the mentioned reference (see Sec. 3.1 of Ref. 22) providing a more accurate classification of the stability points.

ii. $\Lambda>0$

As we shall see, there are two fixed points: $(h_0^\Lambda,0)$ and $(h_1^\Lambda,0)$. From (10) it follows the equation for the fixed points,

$$(3h_i^2-\Lambda)\left[\frac{\gamma}{\delta}\left(h_i^2-\frac{\Lambda}{3}\right)^{1-m}-3h_i\right]=0, \tag{18}$$

which must be solved for different values of m. However, (18) has an obvious solution independent of m, $h_0^\Lambda=\sqrt{\Lambda/3}$, which can be shown to correspond to a saddle point. This solution will be ruled out however since it would imply that the energy density vanishes identically. The other solution h_1^Λ will be analyzed for $m=0,\frac{1}{2},1$, in turn.

- $m=0$

Setting to zero the big square parenthesis in (18) and solving the resulting equation, one obtains

$$h_1^\Lambda=\frac{3\delta}{2\gamma}+\frac{1}{2}\sqrt{\frac{9\delta^2}{\gamma^2}+\frac{4\Lambda}{3}}.$$

We define

$$\Sigma_1^\Lambda=\frac{2-2\Sigma_1}{2\Sigma_1-1}, \qquad \Lambda_0=\frac{27\delta^2}{\gamma^2}\,\frac{1+\Sigma_1^\Lambda}{(\Sigma_1^\Lambda)^2}.$$

For $\Lambda<\Lambda_0$ the fixed point $(h_1^\Lambda,0)$ is an asymptotically stable node. If $\Lambda=\Lambda_0$ the fixed point is an asymptotically stable degenerate node whereas for $\Lambda>\Lambda_0$ it is an asymptotically stable focus.

- $m=1/2$

Now h_1^Λ is given by

$$h_1^\Lambda=\sqrt{\frac{\Lambda/3}{1-9\delta^2/\gamma^2}}.$$

For $0<9\delta^2/\gamma^2<1$, $(h_1^\Lambda,0)$ is an asymptotically stable node whereas for $9\delta^2/\gamma^2>1$ the fixed point is an asymptotically stable focus for any $\Lambda>0$.

- $m=1$

Now $h_1^\Lambda=\gamma/3\delta$. For $\Lambda<\gamma^2/3\delta^2$ we have a saddle fixed point, whereas for $\gamma^2/3\delta^2<\Lambda<\Lambda_1$, where

$$\Lambda_1=\frac{\gamma^2}{3\delta^2}\Sigma_2^\Lambda>0, \text{ with } \Sigma_2^\Lambda=\frac{1}{2a}\left(\gamma+\frac{a}{\gamma}\right)^2+1,$$

the fixed point is an asymptotically stable node. For $\Lambda=\Lambda_1$, it is an asymptotically stable degenerate node whereas for $\Lambda>\Lambda_1$, it is an asymptotically stable focus.

IV. QUALITATIVE ANALYSIS USING THE FULL THEORY

Actually, a proper study of viscous phenomena in the frame of ET requires the use of the full equation (4) (i.e., $b=1$). The physical implications of neglecting the second term of (4) have been analyzed in detail in Refs. 3,13. The use of (4) requires an explicit expression for the temperature T in terms of other variables such as ρ and/or n. So far (with the exception of Ref. 14) the expression adopted for T has been a power-law

$$T=\beta\rho^{r}, \tag{19}$$

where $r\geqslant 0$ and $\beta>0$ are constants, which is the simplest way to guarantee a positive heat capacity.[23] However, we shall see that standard thermodynamic relations restrict the range of r. Cãlvao *et al.*[24] found a general equation for the evolution of temperature when two equations of state,

$$\rho=\rho(T,n), \quad p=p(T,n), \tag{20}$$

are given. However, their equation was obtained in the context of matter creation where Π is reinterpreted as a non-equilibrium pressure associated to particle production. The same equation has been carefully analyzed in Ref. 13, it reads as

$$\frac{\dot{T}}{T}=-\Theta\left[\frac{(\partial p/\partial T)_n}{(\partial\rho/\partial T)_n}+\frac{\Pi}{T(\partial\rho/\partial T)_n}\right]. \tag{21}$$

Obviously, when the equations of state (20) are known the evolution of T is no longer free but fixed by (21). However, only in very few cases these equations are explicitly known, as for instance in the case of a radiation gas or an ideal gas.[25] Equation (19) generalizes in a simple way the Stefan–Boltzmann ($R=1/4$) equation which holds for a radiation-dominated fluid in equilibrium. Thus, in that case we get that both equations of state ρ and p (when a γ-law is used) have T as the only independent variable, i.e., $\partial\rho(p)/\partial T=d\rho(p)/dT$. A useful and interesting relation follows from considering the standard thermodynamic relation[26]

$$\left(\frac{\partial\rho}{\partial n}\right)_T=\frac{\rho+p}{n}-\frac{T}{n}\left(\frac{\partial p}{\partial T}\right)_n, \tag{22}$$

which, by virtue of (3b) and (19), yields

$$r=\frac{\gamma-1}{\gamma}\left(\Rightarrow 0<r<\frac{1}{2}\right), \tag{23}$$

i.e., r is no longer an independent parameter (we are indebted to Roy Maartens for pointing us out this restriction). It has been argued[2] that the inequality $r<1$ is reasonable from a physical point of view, since ultrarelativistic and cold non-relativistic matter have $r=1/4$ and $r\sim 2/3$, respectively.

However, an alternative equation can be used for T instead of (19). It is well-known that a relativistic ideal monoatomic gas is described by the two equations of state $p=nT$ and $\rho=3nT+m^2M$, where m is the mass of the particles and M the zeroth-order moment of the Maxwell–Boltzmann distribution function (we use units $k_B=1$, k_B being the Boltzmann constant). We see that the γ-law (γ constant) is *not* compatible with the equations of state of a monoatomic gas in *equilibrium* except for radiation ($m=0$). In that case we have $n\propto T^3$ and the two equations of state for p and ρ reduce to the Stefan–Boltzmann equation and the γ-law with $\gamma=4/3$.

In the remainder of this section the full viscous transport equation will be analyzed resorting to the two expressions for the temperature mentioned above: a power-law given by (19) and an ideal gas equation for p together with the γ-law defining ρ, i.e.

$$p=nT, \qquad \rho=\frac{nT}{\gamma-1}. \tag{24}$$

Note that now both T and n are independent variables and only in the equilibrium limit the particle number density depends exclusively on the temperature, $n=n(T)$ (see comments above). It must be stressed that the Stefan–Boltzmann equation together with an ideal gas equation of state (with $n \propto T^3$) implies $\Pi=0$. So we conclude that out of equilibrium we are forced to adopt one of the two possibilities: (i) a power-law for T with no dependence on n at all; (ii) an ideal gas equation of state for the pressure together a γ-law, with n an independent variable on the same footing as T. Both approaches will be considered in turn.

A. Potential law for the temperature

Using equations (2), (3), (6), (19) and (23) the equation governing the evolution of the Hubble factor reduces to

$$\ddot{H}+\frac{3}{2}[1+\gamma(1-r)]H\dot{H}+\frac{a}{\delta}\left(H^2-\frac{\Lambda}{3}\right)^{1-m}\dot{H}+\frac{(3H^2-\Lambda)a}{2}$$
$$\times\left[\frac{\gamma}{\delta}\left(H^2-\frac{\Lambda}{3}\right)^{1-m}-3\left(1-\frac{\gamma}{2}\right)H\right]-3(r+1)\frac{H\dot{H}^2}{3H^2-\Lambda}=0. \tag{25}$$

i. $\Lambda=0$

- $m=1/2$

As in Section III only the case with γ and δ fulfilling the restriction

$$\frac{\gamma}{\delta}=\frac{3}{2}(2-\gamma), \tag{26}$$

is physically meaningful. In such instance there exists an infinity of stable fixed points $(h_1,0)$, with h_1 an arbitrary positive real number. The phase portrait are parallel stable straight lines.

- $m\neq 1/2$

There are two fixed points, $(0,0)$ and $(h_1,0)$, in the interval $m \in [0,\frac{1}{2})\cup(\frac{1}{2},2)$, where

$$h_1=\left(\frac{3\delta(2-\gamma)}{2\gamma}\right)^{1/(1-2m)}. \tag{27}$$

For $m\geqslant 2$ only the $(h_1,0)$ fixed point occurs, which we analyze next as nothing new arises about the point $(0,0)$.

Let us define the parameter

$$\Sigma_2=\frac{[\gamma(2-a)+2a]^2}{8a\gamma^2(2-\gamma)}>0.$$

For $m<\frac{1}{2}-\Sigma_2$ the equilibrium point is an attractor in the phase space (asymptotically stable focus). For $\frac{1}{2}-\Sigma_2\leqslant m<\frac{1}{2}$ we have asymptotically stable nodes instead. Finally if $m>1/2$, the fixed point is a saddle.

ii. $\Lambda>0$

From Eq. (25) it follows the equation for the fixed points,

$$(3h_i^2-\Lambda)\left[\frac{\gamma}{\delta}\left(h_i^2-\frac{\Lambda}{3}\right)^{1-m}-3\left(1-\frac{\gamma}{2}\right)h_i\right]=0, \tag{28}$$

where, as in the truncated case, only the solutions vanishing the big square parentheses make sense from a physical point of view. Equation (28) will be solved only for three different values of m. In this case we must take into account the constraint (23).

- $m=0$

The fixed point is $(h_2^\Lambda,0)$ with

$$h_2^\Lambda=\frac{3\delta}{2\gamma}\left(1-\frac{\gamma}{2}\right)+\frac{1}{2}\sqrt{\frac{9\delta^2}{\gamma^2}\left(1-\frac{\gamma}{2}\right)^2+\frac{4\Lambda}{3}}.$$

Defining the two new parameters,

$$\Sigma_3^\Lambda=\frac{2(1-\gamma/2)}{1/a[1-a/2+a/\gamma]^2-2(1-\gamma/2)}-1.$$

and

$$\Lambda_2=\frac{27\delta^2}{\gamma^2}\left(1-\frac{\gamma}{2}\right)^2\frac{1+\Sigma_3^\Lambda}{(\Sigma_3^\Lambda)^2},$$

we see that for $\Lambda\leqslant\Lambda_2$ the fixed point is an asymptotically stable node, whereas for $\Lambda>\Lambda_2$ it is an asymptotically stable focus.

- $m=1/2$

The fixed point is

$$h_2^\Lambda=\sqrt{\frac{\Lambda/3}{1-(9\delta^2/\gamma^2)(1-\gamma/2)^2}},$$

so

$$\delta<\frac{2\gamma}{3(2-\gamma)}.$$

Let us introduce

$$\delta_0=\frac{2\gamma}{3(2-\gamma)}(1-\Sigma_2)^{1/2}.$$

For $\delta>\delta_0$, $(h_2^\Lambda,0)$ is found to be an asymptotically stable node, however if $\delta=\delta_0$ it is an asymptotically stable degenerate node, whereas for $\delta<\delta_0$ the fixed point is an asymptotically stable focus. In the radiation case—i.e. $\gamma=4/3$- $\delta_0<0$ and the fixed point is a stable node.

- $m=1$

In this case

$$h_2^\Lambda=\frac{\gamma}{3\delta(1-\gamma/2)}.$$

Let us define the parameter

$$\Lambda_3=\frac{\gamma^2(1+2\Sigma_2)}{3\delta^2(1-\gamma/2)^2}>0.$$

For $\Lambda > \Lambda_3$ the fixed point is an asymptotically stable focus, and for $\Lambda = \Lambda_3$ an asymptotically stable degenerate node.

Finally, when $\Lambda < \Lambda_3$ we can distinguish two subcases. Defining

$$\Lambda_3^* = \frac{\gamma^2}{3\delta^2(1-\gamma/2)^2},$$

we have a saddle point for

$$\Lambda < \Lambda_3^*,$$

and an asymptotically stable node for

$$\Lambda_3^* < \Lambda < \Lambda_3.$$

B. Ideal gas equation for the temperature

In this section we shall study the specific behavior of the equilibrium points, making use of the state equations (24). As neither particle production nor annilation occurs, n obeys the conservation equation

$$\dot{n} + 3Hn = 0, \tag{29}$$

which leads to $n \propto R^{-3}$. The expression for the temperature,

$$T = \frac{3}{\kappa}\frac{\gamma-1}{n_0}R^3\left(H^2 - \frac{\Lambda}{3}\right), \tag{30}$$

where $n_0 > 0$ is a constant, follows easily. Using (2), (3a), (3b), (4), (6) and (30) we get the equation

$$\ddot{H} + \frac{a}{\delta}\left(H^2 - \frac{\Lambda}{3}\right)^{1-m}\dot{H} + \frac{(3H^2-\Lambda)a}{2}\left[\frac{\gamma}{\delta}\left(H^2 - \frac{\Lambda}{3}\right)^{1-m} - 3\left(1 - \frac{\gamma}{2}\right)H\right] - 6\frac{H\dot{H}^2}{3H^2-\Lambda} = 0, \tag{31}$$

i. $\Lambda = 0$

We have the same fixed points as in the truncated theory [see equation (17)].

In the case $m = 1/2$ the discussion runs along the same lines as that of the truncated theory. After linearizing the system and introducing the parameter

$$\Sigma_3 = \frac{a}{4\gamma^2},$$

the following will be discussed. For $m < \frac{1}{2} - \Sigma_3$ the eigenvalues are complex and the equilibrium point is an attractor (asymptotically stable focus). For $m = \frac{1}{2} - \Sigma_3$ there is a bifurcation point which is an asymptotically stable degenerate node. For $m > 1/2$ one has a saddle point. Finally, if $\frac{1}{2} - \Sigma_3 < m < 1/2$ the fixed point results an asymptotically stable node.

ii. $\Lambda > 0$

Now the fixed points $(h_2^\Lambda, 0)$ are again the same as in the full theory using a power law for the temperature.

- $m = 0$

For any $\Lambda > 0$ the fixed point is an asymptotically stable focus.

- $m=1/2$

Defining

$$\delta_1=\frac{\gamma}{3}\sqrt{1-\frac{1}{2\gamma^2}},$$

we note that if δ lies in the interval $0<\delta<\delta_1$ the fixed point is an asymptotically stable focus. If $\delta=\delta_1$ it is an asymptotically stable degenerate node, and if $\delta_1<\delta<\gamma/3$ an asymptotically stable node for any $\Lambda>0$.

- $m=1$

Let us define

$$\Lambda_4=\frac{\gamma^2}{3\delta^2}(1+2\Sigma_3).$$

If $\Lambda<\gamma/3\delta^2$ then the fixed point is a saddle point, but if $\gamma/3\delta^2<\Lambda<\Lambda_4$ it is an asymptotically stable node. For $\Lambda=\Lambda_4$ it is an asymptotically stable degenerate node, and for $\Lambda>\Lambda_4$ an asymptotically stable focus.

1. Non-thermalizing condition for dissipative de Sitter expansion

i. $\Lambda=0$

From (5) and (2) one finds

$$H^{1-2m}<\delta/a\,. \tag{32}$$

For the truncated and full theory using an ideal gas equation for T this condition reduces to $\gamma>3a$ by virtue of (17). On the other hand, as the velocity of the viscous pulses, as well as the speed of sound, cannot exceed the speed of light $(1<\gamma<2)$ we obtain the restrictions on γ and a. If a lies in the range $0<a<\frac{1}{3}$, the two mentioned conditions amount to $1<\gamma<2$; whereas if $\frac{1}{3}<a<\frac{2}{3}$ these restrictions imply $3a<\gamma<2$. Finally, if $\frac{2}{3}<a<1$, no γ can fulfill both conditions.

For the full theory with a power law for temperature one obtains the restriction $\gamma>\gamma_c$ where

$$\gamma_c=\frac{6a}{3a+2}.$$

Two conditions must be fulfilled simultaneously by γ: $1<\gamma<2$ and $\gamma>\gamma_c$. For $0<a<\frac{2}{3}$, these restrictions imply $1<\gamma<2$ (since $\gamma_c<1$); whereas for $\frac{2}{3}<a<1$ one has $1<\gamma_c<\gamma<2$. So for $a<1$ the full theory with a power law for temperature always holds.

ii. $\Lambda>0$

Instead of (32) we now have

$$\frac{a}{\delta}\left(H^2-\frac{\Lambda}{3}\right)^{1-m}<H, \tag{33}$$

when a positive cosmological constant is present.

For the fixed point $(h_1^\Lambda,0)$ (that of the truncated theory) the restrictions for γ are the same that in the truncated case with $\Lambda=0$, whereas for the full theory the restriction (33), when applied to the point $(h_2^\Lambda,0)$, coincides with that of the full theory using a power law for the temperature with vanishing Λ.

V. DYNAMICAL CONSEQUENCES

In this section we study the dynamical implications of linearizing the equation for H. This linearization allows one to obtain an analytical solution for $R(t)$ near the equilibrium points. The matrix $\mathbf{L}$ is given by (14) and the system of differential equations to solve is

$$\begin{pmatrix} \dot{h} \\ \dot{X} \end{pmatrix} = \begin{pmatrix} P_H & P_X \\ Q_H & Q_X \end{pmatrix} \begin{pmatrix} h \\ X \end{pmatrix}, \tag{34}$$

where $h \equiv H - h_i$ and $\overline{X} \equiv X - X_i = X$ being $(h_i, X_i = 0)$ the fixed points. Equation (34) can be written as

$$\ddot{h} - Q_X \dot{h} - Q_H h = 0, \tag{35}$$

where in our model $P_H = 0$ and $P_X = 1$. The corresponding characteristic equation reads as

$$\lambda \pm = \frac{Q_X \pm \sqrt{Q_X^2 + 4Q_H}}{2},$$

which coincides with the equation for the eigenvalues of $\mathbf{L}$. We perturb the system around the de Sitter solution for $t=0$, i.e., $H(t=0) = h_i + \epsilon(0)$ and take $\dot{H}(t=0) = \dot{\epsilon}(0)$ as initial condition.

A. Saddle points and nodes

In the neighborhood of these points the discriminant Δ is positive and the eigenvalues $\lambda_\pm$ are real and different. For $det(\mathbf{L}) < 0$ one has a saddle point, and for $det(\mathbf{L}) > 0$ a node. The solution of (35) is

$$h = c_1 e^{\lambda_+ t} + c_2 e^{\lambda_- t},$$

with

$$c_1 = -\frac{\dot{\epsilon}(0) - \epsilon(0)\lambda_-}{\lambda_- - \lambda_+}, \quad c_2 = \frac{\dot{\epsilon}(0) - \epsilon(0)\lambda_+}{\lambda_- - \lambda_+}.$$

Upon integration, one has for the scale factor

$$R(t) \propto e^{h_i t} \exp\left[\frac{c_1}{\lambda_+} e^{\lambda_+ t} + \frac{c_2}{\lambda_-} e^{\lambda_- t}\right], \tag{36}$$

which shows superinflationary expansion if initial conditions are taken such that c_1, c_2 are positive. This type of evolution for $R(t)$ on time has been obtained previously in a different context.[27] In this case the fluid when submitted to a small perturbation, goes away from the equilibrium point expanding much more rapidly than the de Sitter's. In the case of nodes, and when $\lambda_+ + \lambda_- = Q_X$ is positive (negative), the node will be unstable (stable).

B. Attractors and repellors

We study here the behavior of the scale factor near a sink (an asymptotically stable attractor) and a source (an asympotically unstable repellor). The solutions of the characteristic equation are complex,

$$\lambda_\pm = \frac{Q_X \pm i\sqrt{|\Delta|}}{2},$$

where $\Delta = Q_X^2 + 4Q_H$. Then the solution of (35) is

$$h = c_1 e^{\frac{Q_X}{2}t} \sin \frac{\sqrt{|\Delta|}}{2}(t + c_2).$$

The integration constants can be determined through the initial conditions. They read as

$$c_2 = \frac{2}{\sqrt{|\Delta|}} \tan^{-1}\left[\frac{\epsilon(0)\sqrt{|\Delta|}}{2\dot{\epsilon}(0) - \epsilon(0)Q_x}\right]$$

and

$$c_1 = \frac{\epsilon(0)}{\sin\,(\sqrt{|\Delta|}/2)c_2}.$$

Integrating the equation for h, one follows that

$$R(t) \propto e^{h_i t} \exp\left[\frac{2c_1}{Q_x^2 + |\Delta|}\left(Q_x \sin\frac{\sqrt{|\Delta|}}{2}(t + c_2) - \sqrt{|\Delta|}\cos\frac{\sqrt{|\Delta|}}{2}(t + c_2)\right)\right]. \tag{37}$$

For $Q_X < 0$ we have an attractor (the scale factor undergoes an oscillatory approach to the de Sitter solution) and for $Q_X > 0$ it is a source, i.e., the scale factor deviates from the de Sitter solution.

C. Degenerate nodes

In this case $\Delta = 0$ and $\lambda_+ = \lambda_- = \lambda = Q_X/2$. The solution of (35) is

$$h = c_1 e^{\lambda t} + c_2 t e^{\lambda t}.$$

Because of the initial conditions the integration constants are

$$c_1 = \epsilon(0), \quad \text{and} \;\; c_2 = \dot{\epsilon}(0) - \lambda\epsilon(0).$$

Integration of the expression for h leads to

$$R(t) \sim e^{h_i t} \exp\left[\frac{e^{\lambda t}}{\lambda}\left(\epsilon(0) + (\dot{\epsilon}(0) - \lambda\epsilon)\left(t - \frac{1}{\lambda}\right)\right)\right], \tag{38}$$

hence $R(t)$ approaches to or separates from the de Sitter solution depending on the sign of λ. The rate of evolution is faster than in the de Sitter case.

D. Energy conditions

The weak energy condition (WEC) states that $T_{ab}W^a W^b \geqslant 0$, where T_{ab} is the energy-momentum tensor given by (1) and W^a a generic timelike vector. In our model this condition reduces to

$$\rho + \Lambda \geqslant 0. \tag{39}$$

The dominant energy condition (DEC) imposes $T_{ab}W^a W^b \geqslant 0$ and $-T^{ab}W_a$ to be a non-spacelike vector which is equivalent to $T_{00} \geqslant |T_{ab}|$.[28] This conditions is fulfilled in our case solely if

$$-\rho \leqslant p + \Pi \leqslant \rho + 2\Lambda. \tag{40}$$

Finally, the strong energy condition (SEC) requires that $T_{ab}W^aW^b+\frac{1}{2}T^a_a\geqslant 0$ which amounts to

$$\rho-2\Lambda+3p+3\Pi\geqslant 0. \tag{41}$$

These conditions can be rewritten in terms of H and $\dot{H}$ as

$$\begin{aligned}&\text{WEC:}\quad H^2\geqslant 0,\\ &\text{SEC:}\quad H^2+\dot{H}\leqslant 0,\\ &\text{DEC:}\quad \dot{H}\leqslant 0 \quad \text{and} \quad 3H^2+\dot{H}\geqslant 0.\end{aligned}$$

As occurs in the standard inflationary scenarios the de Sitter solutions with $\Lambda\geqslant 0$ satisfy the WEC and DEC but not SEC.

VI. CONCLUSIONS

We have carried out a detailed analysis on the stability of de Sitter and static cosmological models, both in the truncated and full theory for the viscous transport equation (with and without a cosmological constant). We have shown that the conditions for the hydrodynamic approach and the nonthermalizing condition in a de Sitter expansion can be fulfilled simultaneously for sufficiently early times. When *no* cosmological constant is considered the stability analysis for the de Sitter solutions leads to similar results in all the cases, i.e., the models are *unstable* only for $m>1/2$. It is remarkable that this result holds for both the truncated and the full version of ET. On the other hand, when a *positive* cosmological constant is included we see that the models can be stable for $m=1$ if Λ is bounded from below. It remains to be proved that this result holds for a generic $m>1/2$ other than 1.

We have stressed the fact that for a radiation gas a different thermodynamic approach exists, depending on whether the particle number density is taken as an independent variable or not. In the case of a power-law for the temperature it exists a relationship between γ and r.

ACKNOWLEDGMENTS

We would like to thank Roy Maartens, Diego Pavón and Winfried Zimdahl for helpful discussions.

This work has been partially supported by the Spanish Ministry of Education under Grant No. PB94-0718. One of us (J.T.) acknowledges financial support from a FPI grant.

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