Search theory aims at identifying optimal strategies that help to promote encounters between a searcher and its target. Statistical physics approaches often identify searchers as random walkers, capitalizing on the idea of search as movement under uncertainty. The assumption that the searcher lacks any information about target locations leads to the fundamental question of how individual paths should be orchestrated. Based on these assumptions, a proper measure of search efficiency is given by the mean-first-passage time (MFPT) of the random walker through the target location, a quantity which is also the focus of interest in many other areas of physics and science [5,6]. Random search theory was spurred in the 1990s as a result of an efficient searcher needs to balance properly the trade-off between the exploration of new spatial areas and the exploitation of nearby resources, an idea which is at the core of scale-free Lévy search strategies. Here we study multiscale random walks as an approximation to the scale-free case and derive the exact expressions for their mean-first-passage times in a one-dimensional finite domain. This allows us to provide a complete analytical description of the dynamics driving the situation in which both nearby and faraway targets are available to the searcher, so the exploration-exploitation trade-off does not have a trivial solution. For this situation, we prove that the combination of only two movement scales is able to outperform both ballistic and Lévy strategies. This two-scale strategy involves an optimal discrimination between the nearby and faraway targets which is only possible by adjusting the range of values of the two movement scales to the typical distances between encounters. So, this optimization necessarily requires some prior information (albeit crude) about target distances or distributions. Furthermore, we found that the incorporation of additional (three, four, ...) movement scales and its adjustment to target distances does not improve further the search efficiency. This allows us to claim that optimal random search strategies arise through the informed combination of only two walk scales (related to the exploitative and the explorative scales, respectively), expanding on the well-known result that optimal strategies in strictly uninformed scenarios are achieved through Lévy paths (or, equivalently, through a hierarchical combination of multiple scales).

An efficient searcher needs to balance properly the trade-off between the exploration of new spatial areas and the exploitation of nearby resources, an idea which is at the core of scale-free Lévy search strategies. Here we study multiscale random walks as an approximation to the scale-free case and derive the exact expressions for their mean-first-passage times in a one-dimensional finite domain. This allows us to provide a complete analytical description of the dynamics driving the situation in which both nearby and faraway targets are available to the searcher, so the exploration-exploitation trade-off does not have a trivial solution. For this situation, we prove that the combination of only two movement scales is able to outperform both ballistic and Lévy strategies. This two-scale strategy involves an optimal discrimination between the nearby and faraway targets which is only possible by adjusting the range of values of the two movement scales to the typical distances between encounters. So, this optimization necessarily requires some prior information (albeit crude) about target distances or distributions. Furthermore, we found that the incorporation of additional (three, four, ...) movement scales and its adjustment to target distances does not improve further the search efficiency. This allows us to claim that optimal random search strategies arise through the informed combination of only two walk scales (related to the exploitative and the explorative scales, respectively), expanding on the well-known result that optimal strategies in strictly uninformed scenarios are achieved through Lévy paths (or, equivalently, through a hierarchical combination of multiple scales).
distant targets—for example if target distribution is patchy and while walking one can have nearby and faraway patches.

While the dynamics in the symmetric regime is thus straightforward to understand in terms of maximization [scenario (i)] or minimization [scenario (ii)] of the area explored, the details driving optimization in the asymmetric regime [scenario (iii)]—in particular, how movement scales determine search efficiency—have remained partially obscure to date [4,14]. This is so because analytical methods for the determination of the MFPT (often valid just for Markovian processes) are difficult to extend to Lévy or other superdiffusive dispersal mechanisms. In the last years, much effort has been devoted to overcome this limitation. For instance, in [15] the asymptotic behavior of the first-passage distribution of Lévy flights in semi-infinite media was obtained. Other authors have derived expressions and scaling properties of MFPTs for moving particles described either by fractional Brownian motion [16,17] or fractional diffusion equations [18–20]. Finally, the alternative approach to approximate Lévy paths through an upper bound truncation (so that Lévy properties hold just over a specific set of scales) has been explored too [14]. But despite these advances, analytical arguments able to explain the different optimization dynamics observed in the asymmetric compared to the symmetric regime are still lacking.

Lévy or scale-free paths can be conveniently approximated through a combination of multiple scales [21]. This is tantamount to expressing power-law functions as a combination of exponentials [22,23] or providing a Markovian embedding for Lévy stochastic processes [24–26]. Composite random walks and, more in general, multiscale random walks (MSRWs) have also emerged recently as an alternative to the presence of scale-free signatures in animal trajectories [22,27]. It is not clear yet whether the emergence of multiscaled movement behavior in biology arises from exploratory behavior tuned by uncertainty (Lévy as the limiting case) [4,14], or else from informed behavioral processes linked to landscape through sensors and/or memory. It is thus important to understand how this multiscaled behavior should be coupled with other relevant landscape magnitudes such as target distributions and searcher-to-target average distances.

Inspired by these ideas, in the present work we derive an exact analytical method for the determination of the MFPT of MSRWs as an approximation to the scale-free case. While the method proposed becomes increasingly complicated as more scales are considered, we show that 2-scale random walks can effectively resolve the exploitation-exploration trade-off emergent in the asymmetric regime by adjusting movement scales to target distances. Furthermore, the comparison between the 2-scale and the 3-scale random walk suggests that incorporating a third scale does not produce any advantage. Therefore, we conclude that an optimal random search strategy in the asymmetric regime consists of combining two informed movement scales that should approximately correspond to nearby/faraway target distances. Hence, an informed adjustment of movement scales improves search efficiency compared to any noninformed strategy (where scales are imposed at random). In the case of noninformed strategies, however, MSRWs approximating the Lévy strategy are the best solution to solve exploitation-exploration trade-offs.

I. DERIVATION OF THE MFPT

We consider for simplicity an isotropic random walk embedded in the one-dimensional finite domain (0, L) with initial position \(x_0\) and absorbing boundaries (so implicitly assuming that surrounding targets are located at \(x = 0\) and \(x = L\)). We will choose \(x_0 < L/2\) arbitrarily, so \(x_0\) can be interpreted as the initial distance of the searcher to the nearest target. The searcher moves continuously with constant speed \(v\) and performs consecutive flights whose duration is distributed according to a multiexponential distribution function \(\varphi(t)\) in the form

\[
\varphi(t) = \sum_{i=1}^{n} w_i \varphi_i(t), \quad \varphi_i(t) = \tau_i^{-1} e^{-t/\tau_i},
\]

so yielding an \(n\)-scale MSRW characterized by the persistence times \(\tau_i\) and their corresponding weights \(w_i\), that satisfy the normalization condition \(\sum_{i=1}^{n} w_i = 1\).

We now define \(\rho_i(x,t;\rho_0)\) as the probability that the walker starts at time \(t\) from \(x\) a single flight characterized by the distribution \(\varphi_i(t)\) [in the following, using a particular distribution \(\varphi_i(t)\) is termed as being in state \(i\)]. The vector \(\rho_0 = \delta(x-x_0)\delta(t-0)(\rho_{10},\rho_{20},\ldots,\rho_{n0})\) accounts for the set of initial conditions in all states, with \(\rho_{i0}\) the probability of being in state \(i\) at \(t = 0\). Using this notation, the multiscale (non-Markovian) walk gets reduced to a set of \(n\) Markovian states which satisfy (according to standard prescriptions of the continuous-time random walk [4]) the mesoscopic balance equations

\[
\rho_i(x,t;\rho_0) = w_i \sum_{k=1}^{n} \int_0^{t} \left( \frac{\rho_{k+} + \rho_{k-}}{2} \right) \varphi_k(t') dt' + \rho_{i0} \delta(x-x_0) \delta(t) \tag{2}
\]

(for \(i = 1,2,\ldots,n\)), where we have introduced the compact notation \(\rho_{i=} \equiv \rho_i(x = \pm v \cdot t - t'; \rho_0)\). The corresponding probability that the walker, passing through \(x\) at time \(t\), is performing at that instant a flight in state \(i\) is given by

\[
P_i(x,t;\rho_0) = \int_0^{t} \left( \frac{\rho_{i+} + \rho_{i-}}{2} \right) \tau_i \varphi_i(t') dt' \tag{3}
\]

Here we have used the relation \(\int_0^{t} \varphi_i(t') dt' = \tau_i \varphi_i(t)\), valid for exponential distributions, which gives the probability that a single flight in state \(i\) will last at least a time \(t\).

Due to the Markovian embedding used, the general propagator of the random walk in an infinite media can be written in the Laplace space (with \(s\) the Laplace argument) as \(P_i(s,x;\rho_0)\) with the probability density for state \(i\), \(P_i(x,t;\rho_0)\), given by a sum of \(n\) exponentials

\[
P_i(x,s;\rho_0) = \sum_{j=1}^{n} \alpha_{ij}(s)e^{-\beta_j(x)x-x_0}/v, \tag{4}
\]

where \(\alpha_{ij}\) and \(\beta_j\) are positive constants to be determined from the solution of the system (1)–(3). Hence, the solution in the interval of interest \((0,L)\) with periodic boundary conditions reads

\[
Q_i(x,s;\rho_0) = \sum_{m=-\infty}^{\infty} P_i(x + mL,s;\rho_0) = \sum_{j=1}^{n} \alpha_{ij}(s) \frac{e^{-\beta_j(x)L-x-x_0}/v e^{-\beta_j(x)x-x_0}/v}{1 - e^{-\beta_j(x)L/v}} \tag{5}
\]
Finally, the exact MFPT can be computed from Eq. (5) by extending known methods for Markovian processes; in particular, we employ here the renewal method for velocity models [28,29]. According to this, we define \( f_i(t; \rho_0) \) as the first-passage-time probability rate for a walker through any of the boundaries while being in state \( i \). The renewal property of Markovian processes allows us then to write the recurrence relations

\[
q_i(t; \rho_0) = f_i(t; \rho_0) + \sum_{k=1}^{n} \int_{0}^{t} f_k(t-t'; \rho_0) q_i(t'; \rho_0) dt',
\]

where \( q_i(t; \rho_0) \) is defined as the probability rate with which the walker hits (not necessarily for the first time) the boundary at time \( t \) while being in state \( i \). The term \( q_i(t; \rho_k) \) has the same meaning but for a walker starting its path at state \( k \) from the boundary (so with \( x_0 = 0 \)). According to (6) the hitting rate \( q_i(t; \rho_0) \) gets divided into those trajectories for which this is the first hitting rate \( f_i(t; \rho_0) \) plus those trajectories that hit the boundary for the first time at a previous time \( t-t' \) in any of the possible \( n \) states [second term on the left-hand side of (6)]. The total first-passage-distribution of the MSRW will read then \( f(t; \rho_0) = \sum_{i=1}^{n} f_i(t; \rho_0) \) where the \( f_i \)'s are to be determined from the system of equations (6), and the general expression for the MFPT will be by definition [4]

\[
\langle T \rangle = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \sum_{i=1}^{n} \frac{df_i(s; \rho_0)}{ds}
\]

Then, to find a closed expression for \( \langle T \rangle \) one just needs to express the hitting rates \( q_i(t; \rho_0) \) and \( q_i(t; \rho_k) \) in terms of the solutions of the random walk (1)–(5). This is given, in analogy to previous works [29,30], by

\[
q_i(t; \rho_0) = \begin{cases} 
\frac{v}{L} Q_0(0; t; \rho_0) , & 0 < x_0 < L, \\
\frac{v}{L} Q_0(0; t; \rho_0) - \delta(t)/2, & x_0 = 0, x_0 = L; \\
\frac{v}{L} Q_0(0, t; \rho_0) - \delta(t)/2, & 0 < x_0 < L
\end{cases}
\]

Here clearly a different behavior for the case when the walker starts from the boundaries is introduced by convenience to make explicit that the walker cannot get trapped by the boundary immediately at \( t = 0 \), but hittings are only possible for \( t > 0 \). In the following we study how the different scales considered in the MSRW contribute to the search efficiency as a function of the two prominent spatial scales present in the problem (i.e., \( x_0 \) and \( L \)).

### A. 1-scale case (\( n = 1 \))

The MSRW scheme described above reduces trivially in this case to a classical correlated random walk (see, e.g., [4,31,32]) for which the free propagator [Eq. (4)] reads

\[
P_1(x, s; \rho_0) = \frac{1}{2v} \sqrt{\frac{s + \tau_1^{-1}}{s}} \exp\left[ \sqrt{s(s + \tau_1^{-1})} |x - x_0| / v \right].
\]

Using the derivations in Eqs. (5)–(8), the MFPT in (7) yields the exact expression obtained by Weiss [33] thirty years ago,

\[
\langle T \rangle = \frac{L}{2v} + \frac{x_0}{v^2 \tau_1}.
\]

So, assuming that \( x_0, L \) are fixed by the external or environmental conditions, we observe that the search optimization of the 1-scale random walk turns out to be trivial: faster searches (i.e., larger values of \( v \)) and straighter trajectories (i.e., \( \tau_1 \rightarrow \infty \)) will monotonically reduce the search time. In particular, note that for \( \tau_1 \rightarrow \infty \) one recovers the result \( \langle T \rangle = L/2v \), which coincides with the result for a ballistic strategy. It is clear then that in 1-scale random walks the exploration-exploitation trade-off \( (L \text{ vs } x_0) \) is always trivially optimized through a ballistic strategy (in agreement with the results in [14]).

### B. 2-scale case (\( n = 2 \))

The exact analytical solution for this case can still be found easily, albeit the general expression for the MFPT obtained is cumbersome; details are provided in Appendix A. A first survey on this solution (which was implemented in MAPLE) allows us to observe that for large values of \( x_0 \) the ballistic-like strategy (i.e., \( \tau_1 \rightarrow \infty, \tau_2 \rightarrow \infty \)) is again the one which minimizes \( \langle T \rangle \). However, for small \( x_0 \) values we find now the emergence of an asymmetric regime in which the optimal is attained for one of the two scales (either \( \tau_1 \) or \( \tau_2 \)) being much larger than the time \( L/v \) required to cover the domain, with the other scale exhibiting a smaller value. The threshold at which this transition occurs (so, the value of \( x_0 \) for which the optimum \( \langle T \rangle \) becomes smaller than \( L/2v \)) turns out to be \( x_0 \approx 0.105L \), a value which is confirmed by random-walk simulations too.

At the sight of these results, we will focus now our interest in providing some limit expressions which can help us to understand how this transition occurs and how the system behaves in the asymmetric regime. First we note that, in solving the exploration-exploitation trade-off, the exploration part will be always optimized through flights much longer than the typical time to cover the whole domain, which explains why one of the two scales (say, \( \tau_1 \)) should be expected to be as large as possible, in particular \( \tau_1 \gg L/v \). Regarding the second scale, the exploitation side of the trade-off (corresponding to exploring the surrounding area searching for nearby targets) should intuitively benefit from choosing a scale of the order of \( \tau_2 \sim x_0/v \), the time required to reach the nearest target. Scales much larger than this would promote exploration instead of exploitation, while scales much smaller would lead to an unnecessary overlap of the searcher’s trajectory around its initial position [14]. Since the asymmetric regime must emerge necessarily from the asymmetric condition \( x_0 \ll L \) we can thus consider that this second scale should satisfy \( \tau_2 \ll L/v \).

Taking the two limits \( (\tau_1 v/L \rightarrow \infty \text{ and } \tau_2 v/L \rightarrow 0) \) into account, our general solution for the MFPT reduces to

\[
\langle T \rangle = \frac{L}{2v} + \tau_2(1 - w_1) \left( 1 - \frac{1 + \frac{Lw_1}{v \tau_1}}{1 + \frac{w_1}{v \tau_2}} \exp \left[ -\sqrt{\frac{w_1}{v \tau_2}} \right] \right)
\]

Visual inspection of this expression already shows that values of the MFPT below the ballistic threshold \( \langle T \rangle = L/2v \) can be now obtained for appropriate combinations of \( x_0, \tau_2, \) and \( w_1 \). In particular, in the limit when \( x_0 \rightarrow 0 \) the previous
expression gets minimized for the value
\[ \tau_1^* = \sqrt{L x_0 w_1 / v^2}, \]
where we use the asterisk to denote values that are optimal. After minimizing \((T)\) also with respect to \(w_1\), we find that the global optimum of the MFPT corresponds to
\[ \tau_2^* = \frac{1}{v} \sqrt{\frac{x_0^2 (L + \sqrt{8L x_0})}{L - 8x_0}}, \quad w_1^* = \frac{2x_0 (L + \sqrt{8L x_0})}{L (L - 8x_0)}. \]

Now, in the limit \(L \to \infty\) we observe that \(\tau_1^* \to x_0 / v\) and \(w_1^* \to 2x_0 / L\). Altogether, these results provide a clear and simple description of the search dynamics in the asymmetric regime for 2-scale MSRWs which confirms our discussion above.

The optimum strategy in the asymmetric regime for 2-scale MSRWs which confirms our discussion above. The optimum strategy does not consist just in an appropriate choice of the time searching intensively its surroundings. So, the optimal strategy does not consist just in an appropriate choice of the scales \(\tau_1\) and \(\tau_2\) but also in using them in an adequate proportion.

Figures 1 and 2 show the comparison between random-walk simulations (symbols) and our method, both for the exact case (solid lines) and for the approximations (11)–(13) (dotted lines). Note in Fig. 1 that the optimum value of the MFPT clearly improves (specially for \(x_0 / L\) very small) the value obtained for a ballistic strategy or a \(\mu = 2\) Lévy walk strategy (dashed and dashed-dotted horizontal lines, respectively), so revealing that an appropriate combination of only two movement length scales can be actually more efficient than a scale-free strategy. The range of validity of the approximated results (11)–(13) is also shown in the plots, as well as the scaling \(\tau_2^* \sim \sqrt{w_1^*}\) derived in Eq. (12) (see Fig. 2).

Despite finding a set of combinations of \(\tau_2\) and \(w_1\) outperforming both Lévy and ballistic strategies, these results show that it is necessary for the searcher to have some information about the domain scales (i.e., \(x_0\) and \(L\)) in order to fine-tune search and get effective strategies. Without this knowledge Lévy or ballistic patterns look like robust strategies, that could be even more effective than searching with badly adjusted movement scales, as suggested by the comparison in Fig. 1. This fact is also confirmed when observing the dependence of \((T)\) on \(\tau_2\) (Fig. 3, and also additional information in Appendix C) in order to assess the range width at which \(\tau_2\) and \(w_1\) lead to optimality. In Fig. 3 we provide the results of our exact solution for different values of \(x_0\) and different weights \(w_1\) (here the approximated results and simulations are not shown in order to facilitate understanding). In accordance to our analytical results above, we observe that values of \(w_1\) close to \(2x_0 / L\) minimize the MFPT. So, there are certain values of \(w_1\) for which the MFPT becomes lower than the ballistic value \(L / 2v\) (but we stress that the most critical parameter for getting below this threshold is clearly \(x_0\)). Actually, for the two upper panels (which correspond to \(x_0 / L = 0.005\) and \(x_0 / L = 0.01\)) we observe that any choice of \(\tau_2\) and \(w_1\) would result in a better (or as good as) performance than a ballistic strategy, while the region where the Lévy strategy is outperformed is relatively small.

We stress finally that we have carried out studies, both analytically and numerically, for the case when the initial position \(x_0\) is not fixed but is distributed according to an exponential or a Gaussian distribution, so a range of \(x_0\) values is allowed (results not shown here). The results for all these cases coincide qualitatively with those reported above, so whenever values \(x_0 < 0.105L\) are predominant the asymmetric regime is
FIG. 3. (Color online) MFPT for a 2-scale random walker with $L = 1000$, $v = 1$, and $\tau_1 = 10^3 L/v$ with different initial conditions. The plot shows the exact analytical solution for different values $w_i = 0.02$ (circles), 0.05 (triangles), 0.1 (inverted triangles), and 0.2 (diamonds). The values obtained for ballistic and $\mu = 2$ Lévy strategies are also given for comparison (dashed and dotted lines, respectively).

recovered. This confirms that the emergence of the asymmetric regime is not an artifact caused by the choice of a fixed initial condition, in agreement with recent numerical studies [34].

C. 3-scale case ($n = 3$)

Provided that the initial time to reach any of the targets is given by the two time scales $x_0/v$ and $(L - x_0)/v$, it could be intuitively expected that these are the only ones necessary to reach an optimal strategy. To check this we have solved analytically the 3-scale case (see Appendix B) and, given that the expression obtained is extremely cumbersome, we have used Markov chain Monte Carlo algorithms in order to determine numerically the global minimum of the MFPT as a function of the parameters $\tau_i$ and $w_i$. The results so obtained are conclusive and confirm the idea that indeed only two scales are needed to minimize the MFPT. We find that for large values of $x_0$ the optimal strategy is again ballistic-like (so only displacements with $\tau_i \gg L/v$ should be performed in order to minimize $\langle T \rangle$). Instead, for $x_0$ small enough the optimum arises through the combination of only two scales which coincide with those found for the optimal 2-scale case; this means that two of the three scales involved (say, $\tau_1$ and $\tau_2$) will eventually have the same value after minimization. Even more surprising is that when the initial condition is governed by two different scales (by combining two different values of $x_0$, each with a given probability) the optimum still corresponds to a 2-scale random walk; in this case the optimum value $\tau_2^*$ is in between the optimum values that one would find for each of the two $x_0$ values alone. Further studies are thus needed to confirm to what extent the combination of only two scales is universally robust and effective enough, independently of the number of prominent spatial scales present in the domain; this point will be the focus of a forthcoming work.

II. DISCUSSION

The main result extracted from the theoretical analysis reported here is that MSRWs with only two movement characteristic scales can represent a mathematical optimum (in terms of MFPT minimization) for random search strategies. One must stress, however, that this just happens in the asymmetric regime, where both nearby and faraway targets are available, while for $x_0$ uniformly distributed a ballistic strategy (which maximizes area exploration) will provide always the trivial optimum. Furthermore, we have proved here that additional (three, four, ... ) scales do not allow us to improve the optimum achieved for 2 scales. The 2-scale solution found is actually able to outperform both ballistic and Lévy strategies but only for specific intervals of the characteristic parameters $\tau_i$ and $w_i$ which depend on the characteristic scales of the domain (namely, $x_0$ and $L$). In particular, the global optimum turns out to be given by $\tau_1 \gg L/v$ and $\tau_2 \approx 2x_0/v$, which can be intuitively justified in terms of optimizing the trade-off between exploring for faraway targets and exploiting nearby resources. While the theoretical analysis provided here has been restricted to the one-dimensional case (for which an exact solution for the MFPT is attainable), we think that these arguments are generally valid and so we expect them to hold in higher dimensions, and probably in more complicated situations as for instance in biased searches too.

In the context of animal foraging, the fact that fine-tuned 2-scale random walks outperform Lévy walks represents a convenient extension of the Lévy flight paradigm from the completely uninformed scenario to that in which domain scales are (partially or completely) available to the organism. In the uninformed case, where the characteristic scales of the search problem are unknown, a scale-free strategy represents a convenient (albeit suboptimal) solution. However, in cognitive systems search optimization programs should be adjustable on the basis of accumulated evidence. As we show here, 2-scale walks would be optimal provided that the searcher has previous available information (at least some crude guess gained from previous experience) about the values of the scales $x_0$ and $L$. Let us stress that we are considering that such prior guess about target distances is limited (by the searcher cognitive capacity) or not informative enough (e.g., landscape noise, insufficient cumulative evidences) to set up a deterministic search strategy. Furthermore, implementation of deterministic rules will become more difficult in more realistic situations for two- or three-dimensional spaces; in those cases it is reasonable to assume that, even if we possess a guess about the scale $x_0$, the random-walk hypothesis is still meaningful. Accordingly, as more and more information about the domain scales becomes integrated by the searcher we should observe a tendency towards a reduction (and an adjustment) in the number (and magnitude) of movement scales used, respectively. This process should go on up to the point where barely one or two scales would persist. Furthermore, we note that for the extreme case of perfectly informed (deterministic) walkers no characteristic search scales at all would be necessary since in that case the search process is plainly directed towards the target.

Our results add then some new dilemmas and perspectives on the fundamental problem of what biological scales could
APPENDIX A: SOLUTION OF THE 2-SCALE COMPOSITE RANDOM WALK

In the \( n = 2 \) case the Eq. (2) from the main text reads

\[
\rho_1(x,t; \rho_0) = \rho_0 \left( \int_0^t \frac{\rho_{11} + \rho_{12}}{2} \psi_1(t') dt' + \int_0^t \frac{\rho_{21} + \rho_{22}}{2} \psi_2(t') dt' \right) + \rho_0 \psi_1(x_0) \psi_1(t), \tag{A1}
\]

\[
\rho_2(x,t; \rho_0) = \rho_0 \left( \int_0^t \frac{\rho_{11} + \rho_{12}}{2} \psi_1(t') dt' + \int_0^t \frac{\rho_{21} + \rho_{22}}{2} \psi_2(t') dt' \right) + \rho_2 \psi_1(x_0) \psi_1(t),
\]

while Eq. (3) becomes

\[
P_1(x,t; \rho_0) = \int_0^t \frac{\rho_{11} + \rho_{12}}{2} \tau_1 \psi_1(t') dt', \tag{A2}
\]

\[
P_2(x,t; \rho_0) = \int_0^t \frac{\rho_{21} + \rho_{22}}{2} \tau_2 \psi_2(t') dt'.
\]

The solution of the system (A1) for an infinite domain can be found, for instance, using standard Fourier-Laplace techniques, and then \( P_1 \) and \( P_2 \) can be conveniently computed through Eq. (A2). Carrying out this analysis, it is not difficult to find that the solution for the probability density of being in state 1 can be written in the Fourier-Laplace space as

\[
P_1(k,s; \rho_0) = \frac{a_1(s) + a_2(s) e^{2k^2}}{b_1(s) + b_2(s) e^{2k^2} + v^4 k^4}, \tag{A3}
\]

where \( s \) and \( k \) are the Laplace and Fourier arguments after transforming in time and space, respectively. The exact expression of the coefficients \( a_i, b_i \) is the following:

\[
a_1 = (s + \tau_1^{-1})^2 (s + \tau_2^{-1}) (s \rho_{11} + \rho_{12} \tau_2^{-1}),
\]

\[
a_2 = \rho_{11} (s + \tau_1^{-1})^2 + (s + \tau_1^{-1}) (s \rho_{10} + w_1 \tau_2^{-1}),
\]

\[
b_1 = s (s + \tau_1^{-1}) (s + \tau_1^{-1}) (s + w_1 \tau_2^{-1} + w_1 \tau_1^{-1}),
\]

\[
b_2 = (s + \tau_1^{-1}) (s + \tau_1^{-1}) (s + \tau_2^{-1}) (s + \tau_2^{-1} w_1).
\]

The Fourier inversion of (A3) leads to an expression of the type

\[
P_1(x,s) = \frac{a_1 - a_2 \beta_-}{2 \beta_- (b_2 - 2 \beta_2)} e^{-\beta_- |x-x_0|/\nu} + \frac{a_1 - a_2 \beta_+}{2 \beta_+ (b_2 - 2 \beta_2)} e^{-\beta_+ |x-x_0|/\nu}, \tag{A5}
\]

in agreement with Eq. (4) from the main text. Here, we introduce the parameters \( \beta_{\pm} \equiv (b_2 \pm \sqrt{b_2^2 - 4 b_3 b_1})/(2 b_3) \), which are the two roots of \( b_1 - b_2 k + b_3 k^2 = 0 \).
The solution of the 2-scale random walk in a finite domain \((0, L)\) with periodic boundary conditions is then given by

\[
Q_1(x, s; \rho_0) = \sum_{m=-\infty}^{\infty} P_1(x + mL, s) = \frac{a_1 - a_2 \beta_-}{2\beta_- (b_2 - 2\beta_-^2)} e^{-\beta_{1,1} s/L} + \frac{a_1 - a_2 \beta_+}{2\beta_+ (b_2 - 2\beta_+^2)} e^{-\beta_{1,1} s/L}.
\]

Note that a solution for \(Q_2(k, s; \rho_0)\) follows now straightforwardly through the exchange \(\tau_1 \leftrightarrow \tau_2, w_1 \leftrightarrow w_2, \rho_{10} \leftrightarrow \rho_{20}\).

The renewal condition used in Eq. (6) from the main text reads in the \(n = 2\) case (after Laplace transform in time)

\[
q_1(s; \rho_0) = f_1(s; \rho_0) + f_1(s; \rho_0) q_1(s; \rho_1) + f_2(s; \rho_0) q_1(s; \rho_2),
\]

\[
q_2(s; \rho_0) = f_2(s; \rho_0) + f_1(s; \rho_0) q_2(s; \rho_1) + f_2(s; \rho_0) q_2(s; \rho_2).
\]

(A7)

Taking into account the MFPT definition we obtain

\[
\langle T \rangle = \lim_{s \to 0} \frac{d}{ds} f_1(s; \rho_0) + f_2(s; \rho_0)
\]

\[
= \lim_{s \to 0} \frac{d}{ds} \left( \frac{q_1(s; \rho_0)[1 + q_2(s; \rho_2) - q_2(s; \rho_1)] + q_2(s; \rho_0)[1 + q_1(s; \rho_1) - q_1(s; \rho_2)]}{1 + q_1(s; \rho_1) + q_2(s; \rho_2) + q_1(s; \rho_1) q_2(s; \rho_2) - q_1(s; \rho_1) q_2(s; \rho_2)} \right),
\]

(A8)

after solving the system (A7).

The MFPT can be then derived by computing the hitting rates through the Laplace transform of Eq. (8) from the main text:

\[
q_1(s; \rho_0) = \frac{v}{b_1} Q_1(0, s; \rho_0) + \frac{a_1,0}{b_1} \frac{2^{b_2,0}}{b_2,0} \left( \frac{2 \tau_1 \sqrt{b_2,0}}{1 - e^{-b_2,0(s-x_0)/v}} \right),
\]

(A9)

\[
q_2(s; \rho_0) = v Q_2(0, s; \rho_0) - 1/2,
\]

(A10)

together with (A6). By doing that we note that in the end the expression of the MFPT will be expressed in terms of the \(s \to 0\) expansion of the coefficients \(a_i\) and \(b_i\) in (A4). In particular, we use the notation \(a_i = a_i,0 + a_i,1 s + a_i,2 s^2 + \cdots\) (and analogously for \(b_i\)) to characterize such expansion. After lengthy calculations and lower order expansion in \(s\) we find

\[
q_1(s; \rho_0) = \frac{a_1,0,0}{b_1,1} s^{-1} + \frac{a_1,0,0}{b_1,1} \frac{2^{b_2,0}}{b_2,0} \left( \frac{2 \tau_1 \sqrt{b_2,0}}{1 - e^{-b_2,0(s-x_0)/v}} \right),
\]

(A11)

The expressions for \(q_2(s; \rho_0)\) will be found again by symmetry through \(\tau_1 \leftrightarrow \tau_2, w_1 \leftrightarrow w_2, \rho_{10} \leftrightarrow \rho_{20}\). Likewise, the form of \(q_1(s; \rho_1)\) (for \(i = 1, 2\)) is found simply by changing the initial condition to the case \(x_0 = 0\) and \(\rho_{10} = 1, \rho_{20} = 0\) [for \(q_1(s; \rho_1)\)] or \(\rho_{10} = 0, \rho_{20} = 1\) [for \(q_1(s; \rho_2)\)]. Replacing all this into (A8) will yield finally the exact MFPT (a compact expression cannot be found except for some particular limits, as stated in the main text).

**APPENDIX B: DERIVATION OF THE 3-SCALE CASE**

The methodology to derive the case \(n = 3\) (as well as for arbitrarily large \(n\)) follows the same lines as in the case shown above. So, the form of the propagator in Fourier-Laplace space for state “1”

\[
P_1(k, s; \rho_0) = \frac{a_1(s) + a_2(s) w^2 k^2 + a_3(s) v^4 k^4}{b_1(s) + b_2(s) w^2 k^2 + b_3(s) v^4 k^4 + v^6 k^6}
\]

allows us to introduce the coefficients \(a_i, b_i\), which have the form

\[
a_1 = (s + \tau_1^{-1})(s + \tau_2^{-1})(s + \tau_3^{-1}) \left\{ \rho_{10} s + \tau_2^{-1}(1 - w_2) + \tau_3^{-1}(1 - w_2) \right\} + \rho_{20} s w_1 (\tau_2^{-1} - \tau_3^{-1}) + \tau_3^{-1}(1 - w_3),
\]

\[
a_2 = (s + \tau_1^{-1}) \left\{ (s + \tau_2^{-1}) \left\{ \rho_{10} (s + \tau_2^{-1}) - \tau_2^{-1}(w_2 + \rho_{20} w_1) \right\} + (s + \tau_3^{-1}) \left\{ \rho_{10} (s + \tau_3^{-1}) - \tau_3^{-1}(w_2 + \rho_{20} w_1) \right\} \right\},
\]

\[
a_3 = \rho_{10} (s + \tau_1^{-1}),
\]

\[
b_1 = s(s + \tau_1^{-1})(s + \tau_2^{-1})(s + \tau_3^{-1}) s^2 \left\{ \tau_1^{-1}(1 - w_1) + \tau_2^{-1}(1 - w_2) + \tau_3^{-1}(1 - w_3) \right\}
\]

\[
- \tau_1^{-1} \tau_2^{-1} w_3 - \tau_1^{-1} \tau_3^{-1} w_2 - \tau_1^{-1} \tau_3^{-1} w_1,
\]

\[
b_2 = (s + \tau_1^{-1})^2(s + \tau_2^{-1})^2 + (s + \tau_1^{-1})^2(s + \tau_3^{-1})^2 + (s + \tau_2^{-1})^2(s + \tau_3^{-1})^2 - (s + \tau_1^{-1})^2(\tau_2^{-1} w_2 (s + \tau_2^{-1}) + \tau_3^{-1} w_3 (s + \tau_3^{-1})),
\]

\[
- (s + \tau_2^{-1})^2(\tau_1^{-1} w_1 (s + \tau_1^{-1}) + \tau_3^{-1} w_3 (s + \tau_3^{-1})),
\]

\[
b_3 = (s + \tau_1^{-1})^2 + (s + \tau_2^{-1})^2 + (s + \tau_3^{-1})^2 - \tau_1^{-1} w_1 (s + \tau_1^{-1}) - \tau_2^{-1} w_2 (s + \tau_2^{-1}) - \tau_3^{-1} w_3 (s + \tau_3^{-1}).
\]

(B2)
On the other side, the MFPT must be written as

\[ (T) \equiv \lim_{s \to 0} \frac{d[f_1(s; \rho_0) + f_2(s; \rho_0)]}{ds}, \quad \text{(B3)} \]

with the first-passage probability rates \( f_i(s; \rho_0) \) coming from the system of equations

\[
\begin{align*}
q_1(s; \rho_0) &= f_1(s; \rho_0) + f_1(s; \rho_0)q_1(s; \rho_1) + f_2(s; \rho_0)q_1(s; \rho_2) + f_3(s; \rho_0)q_1(s; \rho_3), \\
q_2(s; \rho_0) &= f_2(s; \rho_0) + f_1(s; \rho_0)q_2(s; \rho_1) + f_2(s; \rho_0)q_2(s; \rho_2) + f_3(s; \rho_0)q_2(s; \rho_3), \\
q_3(s; \rho_0) &= f_3(s; \rho_0) + f_1(s; \rho_0)q_3(s; \rho_1) + f_2(s; \rho_0)q_3(s; \rho_2) + f_3(s; \rho_0)q_3(s; \rho_3). \\
\end{align*}
\]

The explicit expression for the MFPT will be found, then, after the 3-scale random walk has been solved in Laplace space and the \( s \to 0 \) expansion has been performed at the level of the hitting rates \( q \). All this leads to

\[
q_1(s; \rho_0) = \frac{a_{1,0}v}{b_{1,1}L^{s-1}} + \frac{a_{1,0} - a_{2,0}b_2 + a_{2,0}b_2^2 + a_{2,0}b_2^4 + a_{1,0} - a_{2,0}b_2 + a_{2,0}b_2^4}{2\beta_+ (b_{2,0} - 2b_{3,0}b_2^2 + 3b_2^4)} 1 - e^{-\beta_+ L/v} + \frac{a_{1,0} - a_{2,0}b_2 + a_{2,0}b_2^4}{2\beta_- (b_{2,0} - 2b_{3,0}b_2^2 + 3b_2^4)} 1 - e^{-\beta_- L/v},
\]

where the coefficients \( a_{i,j}, b_{i,j} \) represent, as above, the \( j \)th order term of \( a_i \) and \( b_i \) in the expansion for \( s \) small. Note that we have also introduced in (B5) for simplicity the additional coefficients \( \beta_{\pm} = (b_{3,0} \pm \sqrt{b_{2,0} - 4b_{3,0}})/2 \).

The explicit expressions for \( q_2(s; \rho_0) \) and \( q_3(s; \rho_0) \) could be found just through the exchange \( 1 \leftrightarrow 2 \) and \( 1 \leftrightarrow 3 \) in (B5), respectively. Finally, \( q_1(s; \rho_0) \) will follow also the expression in (B5) just by replacing the initial conditions \( \rho_0 \) conveniently.

**APPENDIX C: BALLISTIC VS LÉVY VS 2-SCALE WALKS**

As stated in the main text, 2-scale random walks are found to provide the global optimum of random searches provided the two scales are chosen conveniently. In order to clarify how sensitive this result is to the choice of the scales, here we complement the picture provided in Figs. 1–3 by showing the results for a fixed weight of the ballistic scale, \( w_1 = 0.02 \). The corresponding plot (Fig. 4) allows us to provide a clear and visual idea of how wide the range of values of \( \tau_2 \) is that the searcher has available to outperform ballistic and/or Lévy strategies. In accordance to the discussion in the main text, we observe that when a small weight is given to the ballistic scale \( \tau_1 \to \infty \) this favors the emergence of a region of \( \tau_2 \) values that, for \( x_0 \) small enough, leads to a MFPT lower than a pure ballistic or a Lévy strategy.