Abstract

In this paper, we analyse a dynamic duopoly investment game for two competing online search engines. Search engines play a vital role for online search and online advertising as they give agents the opportunity to buy and sell products at the precise instant buyers shows interest on a particular product.

This paper aims to identify the key factors that affect the development of market structure, and to explore under which circumstances the leader has incentives to invest more than the laggard, and whether there is increasing dominance. Policy implications will be discussed.

Key words: Search engine, Dynamic competition, Investment, Dominance
Introduction

In this paper we consider a dynamic duopoly investment game for search engines. Internet search engines have become popular in terms of shopping, online advertising media and also as information-seeking vehicles. The aim is to explore a model of dynamic competition between two competing search engines, as for example, Google and Bing. In 2015, Google had 64% of market share in US and 89.3% of the global market. Meanwhile, Bing had 21% of market share in US and 4.7% of the global market. Clearly, the market for search engines exhibits high concentration levels.

Being highly concentrated, the market for online search is quite remarkable. It revenues amount to almost half of all online advertising revenues. Estimates from “eMarketer” indicate that advertising cost worldwide reaches $170 billion in 2015 with search ads accounting for $81 billion. In addition, online search has affected many traditional markets. For instance, online shopping has become an increasingly common staple of life in the 21st century. Online shopping offers a sense of ease and comfort, the ability to scroll quickly through several stores and actual sales, also saves precious time and allows shopping online any time during the day. Search engines play a vital role for shoppers and businesses to buy and sell their products and services.

This paper addresses the following questions: “How market structure in a model of dynamic competition between online search engines evolves over time? Do asymmetries between search engines tend to increase or decrease over time? More specifically, the analysis shall explain whether the leader has incentives to invest more than the laggard, and whether this fact increases the distance between the two firms. The ultimate goal is to determine whether the distance between search engines tends naturally to widen or narrow and to provide some policy recommendations.

We consider a dynamic model in which two firms or search engines, 1 and 2, invest in R&D to improve their search services. In particular, we develop a dynamic duopoly investment game which reflects the following real facts: i) the quality of the search engine’s services improves as it handles more user queries, ii) having a greater query scale allows search engines to innovate faster, iii) search engines will attract advertisers more easily as they get more queries.

We assume that the level of quality of the search algorithm of each firm increases with the number of queries that it receives and its level of investment in innovation in each period. As the level of quality of search services increases with the number of queries and the level of investment in innovation, and as the marginal cost of investing in R&D diminishes with level of

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3 http://www.emarketer.com
quality, firms will have incentives to behave strategically in order to improve their future positions. We characterize the firms’ optimal decisions, predict market outcomes, and determine whether the market for online search exhibits increasing dominance. In particular, we evaluate that firm with higher state variable invests more in equilibrium (weak increasing dominance) and discuss whereby higher investment by leader leads to higher market share (strong increasing dominance)\(^4\).

The paper is organized as follows. Section 1 presents and discusses related literature. Section 2 lays out the basic assumptions for the model. Section 3 characterizes multi-period competition with myopic firms. Section 4 performs simulations and conducts a comparative statics analysis. Section 5 analyses the open-loop equilibrium. Section 6 discusses future research agenda.

1. Literature Review

This paper is related to the literature on market structure and increasing dominance in the search engines market. To the best of our knowledge, this is probably the first attempt to analyse dynamic game in the search engines market. Most of the existing relevant papers focus on properties of the market in the static competition or dynamic competition but in the form of two-sided market which connect indirectly consumers (searchers) and online content providers.

Gandal (2001) analyses competition in the internet search engine market by using two models: growth model and discrete-choice model of product differentiation. He argues that early entrants (Yahoo, Lycos, Excite, Infoseek, and Altavista) can make benefit in the long-run from their first mover position if they continue to innovate and stay ahead in the quality dimensions. In another study, Pereira (2002) evaluates the effect of consumer prices on competition for price-comparison search engines in a partial equilibrium search model. The model has four main characteristics: i) price comparison search engines make profit from vendors, ii) products are branded, iii) vendors can post different prices at their homepages and search engines, iv) vendors inform rival’s prices at a search engine. They conclude that in equilibrium, consumers buy products from homepage of their preferred brand or search engines based on their price and access cost. Mukhopadhyay, Rajan and Telang (2003) analyze the brand effect and first mover advantage in a two period dynamic games. They adopt a vertical differentiation model in order to show that low quality search engines can remain in the market. The model has two main properties: i) consumers are not charged for using online search services; ii) consumer may make more than one query during a single session.

\(^4\) Athey and Schmutzler (2001) discuss weak and strong increasing dominance.
They assume that the incumbent has the brand loyalty advantage and that the entrant enters in the second period and therefore has a cost advantage due to a superior technology. Yet the authors find that the incumbent will emerge as the leader in equilibrium.

Market structure of search engines and implications in terms social welfare are studied by Pollock (2009). To explain the high concentration levels observed in the market, he develops a basic static model which captures the main features of the search engine market. The results imply that market will evolve into monopoly, Google maintain dominant position in many countries, and thus would digress from optimal provision and therefore inefficiently low search quality can appear. Zhao and Tse (2011) develop a two-sided market model in order to study industry structure and competition between search engines. They propose some strategies such as leader should create new two-sided market platform with strong positive cross network effect in order to keep leadership position. Their results imply that a market with a late comer in the search engine market with a superior technology, like Google is characterized by dominance.

Argenton and Prufer (2012) develop an R&D investment model to study competition between search engines. They point out that the search engine market is specified by an inter-temporal type of indirect network externalities. They compare results of a competitive oligopoly market structure with a less competitive current situation. They conclude that if competition between search engines is based on the actual algorithm qualities, then the rate of innovation, quality and total welfare are higher. In another study, Lianos and Motchenkova (2013) develop a model of two-sided market which investment in R&D improves the quality of the search engine. They also analyse market concentration and market dominance. They argue that Google dominates the market at least in the United States and in Europe and suggest that some policies are needed to avoid possible abuse of dominance which may lead to the monopolization of this market.

Kim and Tse (2012) analyze dynamic competition in a model with and without knowledge sharing services between two heterogeneous search engines. They consider a dynamic two-sided market price competition model that connects searchers and information on the Internet. The inferior search engine benefits from having knowledge sharing service. In another study, Kim and Tse (2014) analyse static competition between search engines under the knowledge sharing services. Search engines maximize profit or market share. They obtain that strategies depend on the different search qualities and on the amount of information available on the Internet.

The contribution of this paper is to identify conditions in which asymmetric between search engines tends to increase over time and whether there is increasing dominance. These points differentiate our analysis from much of the existing literatures. As discussed literatures show and our simulations exhibit, in some cases, monopoly could raises in the industry.

2. The Model
There are 2 firms, 1 and 2, each providing a search online service. In each period, a mass of consumers of measure 1 is born. Consumers are uniformly distributed on the segment [0,1], whereas firm 1 is located at 0 and firm 2 is located at 1. Each consumer makes one query in one of the two search engines, which are differentiated à la Hoteling by the parameter $\tau > 0$. The level of quality of firm $i$ at period $t$ is $y^t_i \geq 0$. The gross utility that a given consumer derives from using the search engine $i = 1, 2$ at period $t$ is a function of its level of quality $y^t_i$: $\vartheta(y^t_i)$, with $\vartheta_y \equiv \frac{\partial \vartheta}{\partial y} > 0$.

Search engines do not charge consumers for using their services, so a consumer located at $x$ and using the search engine $i$ with quality $y^t_i$ obtains the utility net of differentiation costs:

$$\vartheta(y^t_i) - |x - x_i| / (2\sigma)$$

where $\sigma \equiv \frac{1}{2\tau} > 0$ measures the degree of substitutability between the two search engines and $x_i \in \{0, 1\}$ is the location of firm $i$. Therefore, the number of queries or market share of firm $i$ at period $t$ is given by:

$$\alpha^t(y^t) = \frac{1}{2} + \sigma \left( \vartheta(y^t_i) - \vartheta(y^t_j) \right) \quad i \neq j \quad \text{and} \quad i, j = 1, 2.$$

with $y^t = [y^t_1, y^t_2]$.

In each period, firms may invest the amount $a^t_i$ in R&D so as to improve their search algorithm. We assume that the investment cost function $\Gamma$ is a strictly convex function of the rate of investment. Furthermore, the marginal cost of investing in R&D decreases in the level of quality achieved by the algorithm, which in turn depends on the total number of received queries and on the amount of investment. Thus, $\frac{\partial \Gamma(y^t_i^{-1}, a)}{\partial a} > 0$, $\frac{\partial^2 \Gamma(0, a)}{\partial a^2} > 0$ and $\frac{\partial \Gamma^2(y^t_i^{-1}, a)}{\partial y^t_i^{-1}, \partial a} < 0$.

We assume that the level of quality of the search algorithm of firm $i = 1, 2$ increases with the number of queries that it receives in each period $\alpha_i$ and with its level of investment in innovation $a^t_i$ as follows:

$$y^t_i = y^{t-1}_i (1 - \lambda) + h(\alpha^t_i(y^{t-1}_i), a^t_i) \quad (1)$$

With $h_\alpha, h_a > 0$. By introducing the parameter $\lambda$, we are imposing that some of the past queries are no longer relevant for the performance of the search algorithm. This assumption captures the effect of “fresh” queries which are only relevant in a given moment.

Profit of firm $i$ in the online advertising, $\pi^t_i$, is an increasing function of market share $\frac{\partial \pi^t_i}{\partial \alpha_i} > 0$: a higher market share means a higher number of queries from which it can bring more online advertising profits. Then, firms invest in R&D so as to maximize its total discounted profits. Therefore, for given $(\mathbf{a^t}, \mathbf{y^t})$ the payoff to firm $i$ in period $t$ is given by:
In each period, firms play a two-stage game with the following timing. First, given \( \mathbf{y}^{t-1} \), each firm \( i = 1, 2 \) chooses its level of investment, \( a_i^t \), thereby incurring cost \( \Gamma(y_i^{t-1}, a_i^t) \); \( \mathbf{a}^t \) and past levels of quality \( \mathbf{y}^{t-1} \) determine the value of \( \mathbf{y}^t \). Second, for these levels of qualities, consumers choose the firm they want to joint to, resulting in market shares \( \alpha^1(y_i^t) \) and \( \alpha^2(y_i^t) \) from which firms make some revenues from online advertising. We omit how firms compete for online advertising and assume that there exists a unique equilibrium to such competition for advertising game and that in this equilibrium firms’ revenue can be represented by function \( \pi^i \).

### 2.1. The Linear-Quadratic Setting

The game has a linear-quadratic formulation if the equation of motion (1) is linear and the (non-discounted) payoff functions are quadratic in the state and controls. Consequently, we assume that the gross utility that a given consumer derives from using the search engine \( i = 1, 2 \) is simply a linear function: \( \vartheta(y_i^t) = \varnothing y_i^t \) with \( \varnothing > 0 \). Thus,

\[
\alpha^i(y^t) = \frac{1}{2} + \tilde{\varnothing} \Delta y_{ij}^t
\]

where \( \tilde{\varnothing} \equiv \varnothing \sigma > 0 \) and \( \Delta y_{ij}^t \equiv y_i^t - y_j^t \). To keep (1) linear we assume that \( h(\alpha^i, a_i^t) = \eta \alpha^i + k a_i^t \) with \( \eta, k > 0 \). That is,

\[
y_i^t = y_i^{t-1}(1 - \lambda) + \eta \alpha_i^t(y_i^{t-1}) + k a_i^t
\]

(2)

Furthermore, we assume the following functional form for the investment cost function:

\[
\Gamma(y_i^{t-1}, a_i^t) = (\bar{c} - cy_i^{t-1}) a_i^t + b \alpha_i^t y_i^{t-1}
\]

where \( \bar{c}, c, b > 0 \). Finally, instantaneous revenue for firm \( i \) is given by:

\[
\pi^i = \alpha^i(y^t) + \rho \frac{\alpha^i(y^t)^2}{2}
\]

with \( \pi^i_{a^i} = 1 + \rho \alpha^i > 0 \) and therefore \( \rho > -1 \). \( \pi^i \) can be rewritten as follows

\[
\pi^i = \frac{4 + \rho}{8} + \left[ \frac{2 + \rho}{2} \tilde{\varnothing} \Delta y_{ij}^t \right] \tilde{\varnothing} \Delta y_{ij}^t
\]

(3)

where

\[
\Delta y_{ij}^t = k (a_i^t - a_j^t) + (1 - \lambda + 2\tilde{\varnothing}) \Delta y_{ij}^{t-1}
\]
3. Multi-Period Competition with Myopic Firms

It will be convenient to distinguish between the case in which investments are strategic substitutes and the case in which they are strategic complements. We have that

\[ \pi_{a_i}^i = \frac{\partial \pi^i}{\partial a_i} \frac{\partial y_i^i}{\partial h} \frac{\partial y_i^i}{\partial a_i} \]

and

\[ \pi_{a_i,a_j}^i = \frac{\partial^2 \pi^i}{(\partial a_i^2)^2} \frac{\partial a_i^i}{\partial y_j^i} \frac{\partial h}{\partial a_j^i} \frac{\partial y_i^i}{\partial h} \frac{\partial a_i^i}{\partial a_i^i} = - \frac{\partial^2 \pi^i}{(\partial a_i^2)^2} \left( \frac{\partial y_i^i}{\partial y_j^i} \frac{\partial y_j^i}{\partial a_j^i} \right) \left( \frac{\partial h}{\partial a_i^i} \frac{\partial h}{\partial a_j^i} \right) \]

Note that \( \pi_{a_i,a_j}^i < 0 \) if \( \frac{\partial^2 \pi^i}{(\partial a_i^2)^2} = \rho > 0 \). Since \( \Gamma \) does not depend on \( a_j \), we have the following:

**Lemma 1** Investments are strategic substitutes, \( \pi_{a_i,a_j}^i < 0 \), if profit in the online advertising market is convex with respect to market share \( (\rho > 0) \), and strategic complements, \( \pi_{a_i,a_j}^i > 0 \), if it is concave \( (\rho < 0) \).

We can now give conditions under which there exists weak increasing dominance: investments in equilibrium are higher in those firms with higher state variables, i.e., for \( i \neq j = 1, 2 \), \( y_i > y_j \) implies \( a_i^* (y) \geq a_j^* (y) \).

Regarding the case in which investments are strategic substitutes, the second-order condition requires that

\[ \Pi_{ii} = \frac{\partial^2 \pi^i}{(\partial a_i^2)^2} = \rho \bar{\theta}^2 k^2 - b < 0 \]

From the first order condition \( \frac{\partial \pi^i}{\partial a_i} = 0 \) we obtain reaction curves for each period \( t \):

\[ a_i^t = R^i (a_j^t) = - \frac{1}{\Pi_{ii}} \left[ \left( 1 + \frac{\rho}{2} \right) \bar{\theta} k + \left( (1 - \lambda + 2\eta \bar{\theta}) \Delta y_j^{i-1} - k a_j^t \right) \rho \bar{\theta}^2 k - (\bar{c} - cy_j^{i-1}) \right] \]

Therefore,

\[ \frac{dR^i (a_j^t)}{da_j^t} = \frac{1}{\Pi_{ii}} \rho \bar{\theta}^2 k^2 = \frac{\rho \bar{\theta}^2 k^2}{\rho \bar{\theta}^2 k^2 - b} \]

Stability requires that \( \frac{dR^i (a_j^t)}{da_j^t} > -1 \), which holds iff

\[ 2\rho \bar{\theta}^2 k^2 - b < 0. \]

Equilibrium R&D levels are
\[ a_t^* = -\frac{1}{b(2\rho \tilde{\rho}^2 k^2-b)} \left[ \left( (1 - \lambda + 2\eta \tilde{\rho})b\Delta y_{ij}^{t-1} - kcY^{t-1} \right) \rho \tilde{\rho}^2 k + bc y_i^{t-1} - (b - 2 \rho \tilde{\rho}^2 k^2)\tilde{c} - (2 \rho \tilde{\rho}^3 k^3 - b \tilde{\rho} k)(1 + \frac{\rho}{2}) \right], \]

where \( Y^{t-1} \equiv y_1^{t-1} + y_2^{t-1}. \)

We have that

\[
\frac{\partial a_t^*}{\partial y_i^{t-1}} = -\frac{1}{b(2\rho \tilde{\rho}^2 k^2-b)} \left[ (1 - \lambda + 2\eta \tilde{\rho})b \rho \tilde{\rho}^2 k - c(\rho \tilde{\rho}^2 k^2 - b) \right].
\]

Thus, assuming stability condition holds we can state

\[
\text{sign} \left\{ \frac{\partial a_t^*}{\partial y_i^{t-1}} \right\} = \text{sign} \left\{ (1 - \lambda + 2\eta \tilde{\rho})b \rho \tilde{\rho}^2 k - c(\rho \tilde{\rho}^2 k^2 - b) \right\}
\]

Note that if investments are strategic substitutes \((\rho > 0)\), second-order condition implies weak increasing dominance \((\frac{\partial a_t^*}{\partial y_i^{t-1}} > 0)\). However, in the case of strategic complements the right-hand side of the above expression must be positive. These results are summarized in the following Proposition:

**Proposition 1** If investments are strategic substitutes \((\rho > 0)\), then the model exhibits weak increasing dominance. If investments are strategic complements \((\rho < 0)\), then the model exhibits weak increasing (resp. decreasing) dominance if \(c(b - \rho \tilde{\rho}^2 k^2) > (<) \left| (1 - \lambda + 2\eta \tilde{\rho})b \rho \tilde{\rho}^2 k \right| \).

4. Simulations

This section conducts symmetric and asymmetric simulations for myopic firms. Indeed, we are concerned the comparative statics analysis. We present the parameter values for each simulation and obtain preliminary results.

4.1. Symmetric Simulation

We consider that the quality of each search algorithm is equal to 1 in the first period. The basic assumptions of the model for myopic firms are maintained. Table 1 displays the value of the parameters that are used in symmetric simulations.

Simulations below show that the firm’s payoff decreases over time whereas R&D investment and quality increase over time. Industry converges to the steady state in the long run equilibrium (See figures 1, 2, 3).
Table 1: Value of the parameters in the symmetric case

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
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<tbody>
<tr>
<td>$y_1 = y_2$</td>
<td>1</td>
</tr>
<tr>
<td>$\tilde{c}_1 = \tilde{c}_2$</td>
<td>1</td>
</tr>
<tr>
<td>$c_1 = c_2$</td>
<td>0.2</td>
</tr>
<tr>
<td>$\rho$</td>
<td>1</td>
</tr>
<tr>
<td>$b$</td>
<td>5</td>
</tr>
<tr>
<td>$k$</td>
<td>0.75</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.15</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Next, we conduct a comparative statics analysis with respect: “$\tilde{c}$”, “c”, “$\rho$”.

4.1.1. “$\tilde{c}$” is decreased: Figures 4, 5, 6, 7 disclose the effect of decreasing “$\tilde{c}$” on the industry. As figures show, R&D investment, quality and investment cost are upper but payoff is lower. If “$\tilde{c}$” is increased, then the impact is the opposite as expected.
4.1.2. "c" is decreased: Under decreasing “c”, R&D investment, quality and investment cost are lower but payoff is upper. Figures 8, 9, 10, 11 demonstrate the effect of decreasing “c” on the industry. If “c” is increased, then the impact is the opposite as expected.
4.1.3. “ρ” is decreased: In the case of “ρ” is positive, reduce of “ρ” generates less R&D investment, investment cost and quality over time, though it generates higher industry’s payoff. (See figures 12, 13, 14, 15).

4.1.4. “ρ” is increased: In the case of “ρ” is positive, raise of “ρ” generates more R&D investment, quality, investment cost and less payoff over time. (See figures 16, 17, 18, 19).
Informally, we can conclude that industry may converge to the steady state in the long run equilibrium.

We summarize the impact of the parameters on the interested variables such as R&D investment, quality, investment cost and payoff of firms. Table 2 shows briefly the results.

Table 2: Comparative statics on parameters

<table>
<thead>
<tr>
<th>R&amp;D</th>
<th>Quality</th>
<th>Investment cost</th>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{c} ) (decreased)</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>( c ) (decreased)</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>( \rho ) (decreased)</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>( \rho ) (increased)</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

4.2. Asymmetric Simulation:

So far, we assumed that the firms have identical quality. We now consider an asymmetric setting in which the starting level of quality of the two search engines differ. Table 3 presents the value of the parameters that are used in asymmetric simulations.
Our simulations show that the firm with higher quality has higher R&D investment, investment cost, market share and payoff. As below figures disclose, in the benchmark setting the gap between firms tend to decrease over time and the industry converges to the steady state in the long run equilibrium. (See figures 20, 21, 22, 23, 24, 25)

Table 3: Value of the parameters in the asymmetric case

<table>
<thead>
<tr>
<th>Parameters</th>
<th>values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>1</td>
</tr>
<tr>
<td>$y_2$</td>
<td>0</td>
</tr>
<tr>
<td>$c_1 = c_2$</td>
<td>1</td>
</tr>
<tr>
<td>$c_1 = c_2$</td>
<td>0.2</td>
</tr>
<tr>
<td>$\rho$</td>
<td>2.87</td>
</tr>
<tr>
<td>$b$</td>
<td>5</td>
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<tr>
<td>$k$</td>
<td>2.5</td>
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<tr>
<td>$\lambda$</td>
<td>0.8</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.07</td>
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<tr>
<td>$\phi$</td>
<td>0.3</td>
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</table>
We now conduct a comparative statics analysis with respect to the parameters: “$c$”, “$r$”, “$p$”.

4.2.1. “$c$” is decreased: As competition matures, R&D investment and quality of firms are increased, though payoff is decreased. None of the firms gain market share from decreasing “$c$”. (See figures 26, 27, 28, 29)
4.2.2. “c” is decreased: As figures 30, 31, 32, 33 indicate, R&D investment, quality, market share and payoff of firm1 decrease while they have opposite effect on firm2. In contrast, when “c” is increased the impact is the opposite as expected. Furthermore, distance between two firms decreases and convergence is happened faster.

4.2.3. “ρ” is decrease: Results with respect to decreasing “ρ” is similar to the last case but more intense competition lead to convergence happens later in the industry.
4.2.4. “ρ” is increased: In this case, firm, which is the leader in terms of quality, invest in R&D more intensively than the rival, as a result there is weak increasing dominance. Asymmetry between firms increase over time and the monopoly could arise in the industry.

In general, simulations determine that in some cases such as: “c” is decreased; “c” is decreased; “ρ” is decreased, the laggard invests in R&D more to catch up leader but for the case
that “ρ” is increased, leader has incentives to invest more than the laggard in order to increase
dominance and persists as monopolist in the market. Informally, we can conclude that in the
asymmetric case, if ρ increases (ρ > 0), there is strong increasing dominance. To prove
analytically this result, we will provide conditions under which this happen.

Table 4 shows briefly the impact of the parameters on the interested variables such as R&D
investment, quality, market share and investment cost of firms.

<table>
<thead>
<tr>
<th>Table 4: Comparative statics on parameters</th>
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<tbody>
<tr>
<td>R&amp;D</td>
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<tr>
<td>Firm 1</td>
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<tr>
<td>+</td>
</tr>
<tr>
<td>c (decreased)</td>
</tr>
<tr>
<td>ρ (decreased)</td>
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<tr>
<td>ρ (increased)</td>
</tr>
</tbody>
</table>

4.3. Main Results

We obtain preliminary results of the symmetric and asymmetric simulations for myopic firms. 
We will analytically prove them in the next version of the paper. Results are summarized below:

Statement 1: Industry may converge to the steady state in the long run equilibrium.

Statement 2: Increasing dominance with monopolization is possible if ρ is sufficiently high.

5. Open-Loop Equilibria

Letting β ∈ [0, 1] be the firm’s discount parameter, the firm’s payoff functions are

\[ V^i = \sum_{t=0}^{\infty} \beta^t (\pi^i(a^t, y^{t-1}) - \Gamma(y^{t-1}, a^t)) , \]

where \( a^0 = a_0 \) and \( y^{-1} = y_0 \) are the initial conditions of the industry. The choice in period \( t \) of
firm \( i \)’s R&D investment level, \( a^t_i \), will affect its payoff in period \( t \), but also in period \( t+1 \)
(though the accumulated level of quality, \( y^t_i \)). Thus, the first order condition for all periods \( t \) are

\[ \frac{\partial}{\partial a^t_i} (\pi^i(a^t, y^{t-1}) - \Gamma(y^{t-1}, a^t)) + \beta \frac{\partial}{\partial a^t_i} (\pi^i(a^{t+1}, y^t) - \Gamma(y^t, a^{t+1})) = 0, \]  \tag{4}

where \( y^t_i \) depends on \( a^t_i \) and is given by equation (2). The above difference equation determine
the equilibrium path of firm \( i \). The equilibrium correspondence is the set of equilibrium points
for any \( a^{-1}, a^{t+1} \) and is denoted by \( \psi(a^{-1}, a^{t+1}) \). Thus, \( x^t = (a^0, a^1, \ldots) \) is an open-loop
equilibrium point iff \( a^* \in \psi(a^{t-1}, a^{t+1}) \). Existence of at least one equilibrium path can be ensured if the strategy sets are bounded independently of \( t \) and for any \((a^t, a^{t-1})\),
\[
\bar{\Pi}^i(a^t_i, a^{t-1}_i) \equiv \bar{\Pi}^i(a^t_i, a^{t-1}_i) - \Pi(i, y^{t-2}(1 - \lambda) + \eta a^t + k a^{t-1})
\]
is bounded and concave with respect to \((a^t_i, a^{t-1}_i)\). [TCB: provide conditions for the general and linear-quadratic case.]

The steady state must satisfy equation (4) and the conditions \( a^t = a_s \) and \( y^t = y_s \) for all \( t \), i.e.,
\[
(\bar{\Pi}^i_{a_i} (a_s, a_s) - \bar{\Pi}^i_{a_i} (a_s, a_s)) + \beta (\bar{\Pi}^i_{y_i} (a_s, a_s) - \bar{\Pi}^i_{y_i} (a_s, a_s)) \frac{\partial y^t}{\partial a^i_t} = 0,
\]
(5)

Where \( y_s \) can be obtained inserting \( y_i^{t-1} = y_i^* \) into equation (2):
\[
y^*_i = \frac{\eta}{2} + \frac{\eta k (a_s^2 + a_s^2) - \lambda k a^2_s}{\lambda(2\eta k - \lambda)}. 
\]
The steady state curve for firm \( i \) is then the set of points \( a_s \) that satisfies the above first-order condition; if the value of the steady state is the same for all firms \((a_t^i = a_s \text{ for all } t \text{ and for all } i)\), then the steady state is symmetric. Since the steady state curve for firm \( i \) is linear in \( a_s^2 \) and \( a_s^2 \), there is a unique steady state, which is symmetric:
\[
a_s = \frac{(1 + \beta)(\phi k (1 + \beta) - \beta k (1 - 2\eta)) - (\lambda - c^2)(\lambda - c^2)}{b(a^2 + c^2)}. 
\]

6. Future Research Agenda

In this study which is preliminary version of the paper, we consider a dynamic duopoly investment game for search engines in order to identify the key factors that affect the development of market structure. Informally, we explore particular circumstances that the leader has incentives to invest more than the laggard. In the next step, we are going to prove them analytically. This paper opens a series of questions regarding the results of simulations which we address them in the next version: “Whether dynamic competition for myopic firm goes to steady state of open loop in the long run equilibrium? Which factors work for and against the leader making the greater profit?” In addition, we make a model for monopoly case and draw policy recommendation to avoid possible abuse of dominant such as reduce the quality of search services. Furthermore, we simulate the open-loop equilibrium and show analytically the results.
References